ROBUST STABILITY AND FEEDBACK STABILIZATION CRITERIA FOR SYSTEMS WITH TIME-VARYING DELAYS

BY

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THESIS ABSTRACT

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Systems with time delay can be found in many fields. Time delay may affect systems' performance and stability, and tools are needed to detect the stability of systems with time delay. Many tools have been developed in the frequency and the time domains. One of the most powerful tools is the Lyapunov-Krasovskii method which can be applied for general systems and different delay types. A special version of that method has been used by many researchers in the recent years. That version establishes sufficient conditions for the systems' stability, and the recent research in that direction tried to reduce the conservatism of the developed method. In the present thesis some theorems for different types of delay are developed, which give less conservative results than those previously reported. Moreover, the present methods are extended to design feedback controllers, which ensure robustness against disturbances. All the developed theorems are compared with the recent results, and verified through simulations.

ملخص الرسالة

الاسم: علي ياسين الريح

عنوان الرسالة: معايير الاستقرار و الاستقرار ية المكينة للنظم المصحوبة بالتأخير الزمني

التخصص: هندسة النظم

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النظم المصحوبة بتأخير زمني هي نظم لها وجودها في كثير من المجالات. هذا التاخير الزمني قد يؤثر في اداء واستقرار النظم لذا كانت هناك حاجة لايجاد نظريات لفحص استقرار هذه النظم . كثيراً من الطرق تم اقتراحها لفحص استقرار النظم ذات التأخير الزمني في كلا من النطاق الترددي و الزمني, لكن من اهم الطرق طريقة ليابيونوف كراسوفسكي حيث يمكن ان تستعمل مع مختلف انواع النظم و مختلف انواع التاخير. مؤخرا, اغلب الباحثين صاروا يستخدمون نوعا خاص من هذه الطريقة و هذا النوع الخاص يعطي فقط شروطا كافية لاستقرار النظم ذات التاخير الزمني. كل الابحاث في هذا السياق ترمي لتقليل تحفظ نتائج الطريقة المقدمة. ولقد تم في هذا البحث تطوير نظريات لمختلف انواع التأخير الزمني , هذه النظريات اعطت نتائج اقل تحفظا من جميع الطرق المطورة حديثا علاوة على ذلك فان الطرق التي قدمت في هذا البحث تم تمديدها لتمكن من تصميم و اختيار متحكمات تغذية عكسية و لضمان احتواء النظام للقلاقل. و من أجل اظهار ميزات الطرق المقدمة تم عمل مقارنات لها مع البحوث الاكثر حداثة اضافة لاجراء محاكاة لبعض النظم لتأكيد النتائج المتحصل عليها.

CHAPTER 1

INTRODUCTION

1.1 Introduction

In real life, many systems and phenomena have the property that the future evolution of their states is affected by their previous values. This is called time delay effect or, simply, time delay. Time delay complicates the system analysis and, in some cases, it may affect the system behavior and performance. It turns out that delays are, perhaps, the main causes of instability and poor performance in dynamic systems. Time delay is frequently encountered in various engineering and physical systems [5, 10, 27]. A system with time delay can be defined as a system whose future state values depend, not only on the present, but also on the history of the system [1]. In the literature, this phenomenon has many names, e.g. systems with aftereffect, systems with time lag, and hereditary systems. Such systems are often described by functional differential equations. *A functional equation* is an equation involving a function for different argument values [1]. The retarded functional differential equation is a function of the previous values of the variables.

Systems with time delay can be found in many fields such as: mechanics, physics, chemistry, biology, medicine, economics and communication [1]. The wide appearance of the aftereffect is a reason to consider it as a universal property of the surrounding world. The effect of the time delay can be neglected in some systems if it is not affecting the system behavior and performance, whereas it should be considered in others. In this chapter, time delay systems are discussed with examples from different fields.

1.1.1 Simple Examples of Time Delay Systems

To explain how the delay can affect the system performance and stability, consider the following example of a simple system with a single state:

$$\dot{x}(t) = -ax(t), \qquad a \geq 0 \tag{1.1}$$

This system is stable since the root of the characteristic equation is $-a$. Now, assume a constant time delay is introduced in the system. Consider the equation:

$$\dot{x}(t) = -a\, x(t) + b\, x(t-\tau) \tag{1.2}$$

for a constant delay τ. The system's characteristic equation becomes:

$$s + a - b\, e^{-s\tau} = 0 \tag{1.3}$$

This equation has an infinite number of roots [14], [15], and it is not possible to obtain all of them. However, if $a = 1$ and $b = (\frac{1+\tau}{\tau}e)$ then $s = \frac{1}{\tau}$ is a root for this equation, which means that the system is no longer stable and the delay causes the instability.

Another example for the delay effect in a feedback loop is as follows [1]:

$$\dot{x}(t) = u(t) \tag{1.4}$$

where $x(t)$ is a scalar state and $u(t)$ is the input. The input is given by:

$$u(t) = -k(x(t)) \quad k > 0 \tag{1.5}$$

To check the stability of the system, the following Lyapunov function can be used:

$$V(x) = \frac{1}{2}(x(t))^2 \tag{1.6}$$

The time derivative of $V(x)$ is:

$$\dot{V}(x(t)) = -k(x(t))^2 \tag{1.7}$$

It is clear that $\dot{V}(x(t))$ is always negative. Now, consider the case whereby a delay is introduced in the input. In this case, the closed loop system equation becomes:

$$\dot{x}(t) = u(t-\tau) = -k(x(t-\tau)) \tag{1.8}$$

where τ is the time delay. Figure 1.1 shows the system's response starting at some initial condition for different τ and k. From the figure, it is clear that the system is unstable when $k = 2$ and $\tau = 1$. This happens because the input was designed to decrease the value of $V(x(t))$ with time. For $\dot{V}(x(t))$ to be negative, $u(t)$ should have the opposite sign of $x(t)$. However, the time delay gives both of them the same sign. Hence, $\dot{V}(x(t))$ will be positive and $V(x(t))$ will increase with time, thereby leading

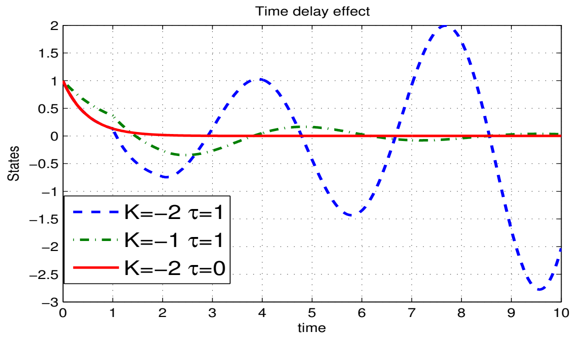

Figure 1.1: Time delay destabilizing effect

to instability of the system.

1.2 Sources of Delay

There are many factors that lead to the appearance of delay in systems. In some cases, delay is inherent in the system's nature, e.g., a period of time is required in an internal combustion engine to mix the air and the fuel. This time is a form of time delay. Another source for time delay is the material transport delay. For example, some time is required for material to travel through a system in heat or mass transfer. Delay also may occur due to the communication among the system parts. For example, time is needed for signals to travel between controllers, sensors and actuators in any typical closed-loop system. Some controllers produce time delay, e.g. in the standard *PID* controller, time delay may be introduced in the system dynamics due to the *I* part of the *PID* controller. Since this part accumulates the error from past values, it is a function of the delayed states. In some cases, the delay is deliberately introduced in the system to attain such goals as quenching the overshoot.

In sequel, delay sources and examples are discussed in more details (see [1]).

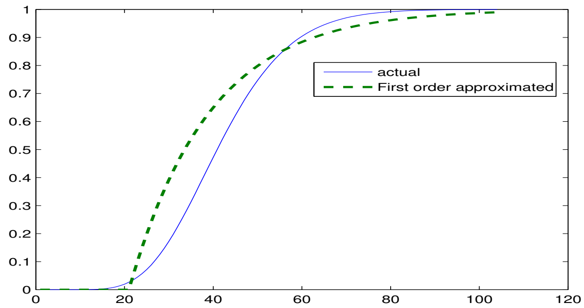

Figure 1.2: Time delay due model approximation

1.2.1 Model Approximation

Consider a system with the transfer function:

$$G(s) = \frac{1}{(s+1)^{10}} \tag{1.9}$$

This system has a relatively higher order. The step response of this system is shown in Figure 1.2. This step response appears similar to that of a first or second order system with time delay. Consider now the following approximation:

$$G(s) = \frac{R_{ss}e^{-hs}}{(\tau s + 1)} \tag{1.10}$$

where the parameters R_{ss}, h and τ should be selected to give the best curve fitting. The response of this approximated model is also shown in Figure 1.2. Here, the two curves are close to each other, and the approximated model can be used to take advantage of its simplicity. Further improvements can be obtained by approximating the function with a higher order model like a second order model with time delay. This type of approximation can be found in some systems. For instance, the process of reactive ion etching has a very complicated model that can be simplified by using this method.

1.2.2 Process Nature

A system's nature may introduce time delay. For example, in chemical reactions some time is required for the reaction to complete, and this required time represents a delay in the system. Another example is the combustion in diesel engines. The diesel fuel is directly injected into the cylinder to be mixed with heated air. The diesel droplets are heated to vaporization. Then, these two components are mixed. The produced mixture starts burning by self-ignition. The time taken for the physical and chemical transformations of the fuel and air mixture to occur before the combustion starts is known as the ignition delay.

1.2.3 Transport Delay

In the systems that contain materials' transfer, time is needed for these materials to travel from one place to another. When a controller is used to control the material characteristics, time delay appears in the response. Such a delay is called transport delay. Consider the following examples:

Rolling mill: In Figure 1.3, two rollers control the width of the passing metal, and a motor adjusts the distance between the rollers. The width is measured by a width sensor, which is placed far away from the rollers because of the high temperature of the metal passing between the rollers. According to the measured width, the distance between the rollers is adjusted. The distance between the rollers' position and the sensor d is related to the velocity of the moving metal $V(t)$ and the time for the metal to reach the sensor τ by the following equation:

$$d = \int_{t-\tau(t)}^{t} V(s)ds \tag{1.11}$$

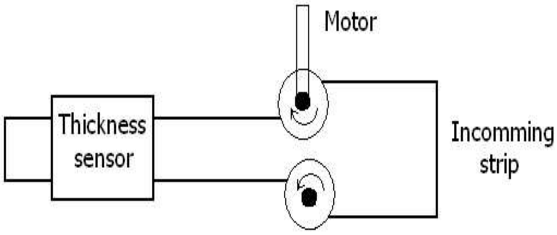

Figure 1.3: Metal rolling system

By taking the time derivative we get:

$$v(t) - (1 - \dot{\tau})v(t - \tau) = 0 \tag{1.12}$$

From the mass conservation law, the thickness $x(t)$ and $v(t)$ are related by:

$$x(t)v(t) = constant \tag{1.13}$$

then,

$$\frac{1}{x(t)} - \frac{1 - \dot{\tau}}{x(t - \tau)} = 0 \tag{1.14}$$

By applying the following integral controller:

$$\begin{aligned} \dot{z}(t) &= x(t - \tau) - x_d, \\ u(t) &= k_i z(t) \end{aligned} \tag{1.15}$$

where x_d is the desired thickness, the system model becomes:

$$\begin{aligned} \dot{\tau} &= \frac{-x(t - \tau) + x(t)}{x(t)} \\ \dot{x}(t) &= k_i(x(t - \tau) - x_d) \end{aligned} \tag{1.16}$$

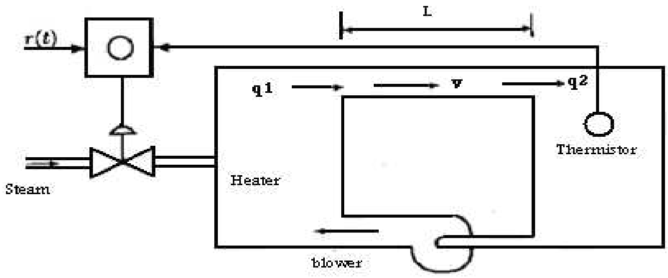

Figure 1.4: Room heating system

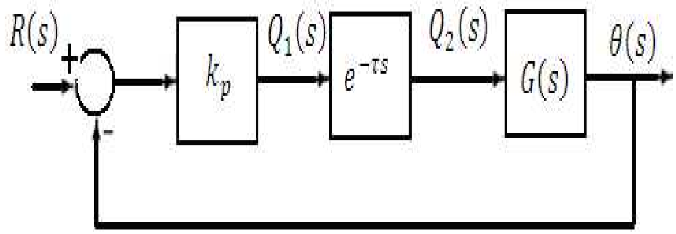

Figure 1.5: Room heating system block diagram

Heating system: In this example, a heater controls the temperature in a room as shown in Figure 1.4. Let $q_1(t)$ be the flow rate of the heat produced from the heater and $q_2(t)$ be the flow rate of the heat entering room. Because of the transportation delay, $q_1(t) = q_2(t - \tau)$ where τ is the propagation time from the heater to the room. The relation between the temperature in the room $\theta(t)$ and $q_1(1)$ is given by:

$$G(s) = \frac{\theta(s)}{Q_2(s)} = \frac{k}{Ts + 1} \tag{1.17}$$

From Figure 1.5, the system model becomes:

$$\dot{x}(t) = Ax(t) + A_d x(t - \tau) + Bu(t - \tau) \tag{1.18}$$

1.2.4 Communication Delay

Generally, the communication delay is of two types: propagation time and access time. The propagation time is the time required by the signal to travel between the actuators, controllers and sensors. Although the signal transmission is considered very fast, in some cases the introduced delay effect cannot be neglected. For example, in guided rocket systems, the communication delay should be considered, as even the smallest communication delay can be intolerable. A satellite controlled from an earth station is another example. The satellite has the coordinates x_1, x_2 and x_3. The radius of the satellite orbit is given by:

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2} \tag{1.19}$$

where m is the satellite mass, and u_1, u_2 *and* u_3 are the control forces applied by thruster jet along each coordinate direction. The satellite dynamics can be described by the following equations:

$$\begin{aligned}
\ddot{x}_1(t) &= -\frac{kx_1(t)}{r^3(t)} + \frac{u_1(t)}{m} \\
\ddot{x}_2(t) &= -\frac{kx_2(t)}{r^3(t)} + \frac{u_2(t)}{m} \\
\ddot{x}_3(t) &= -\frac{kx_3(t)}{r^3(t)} + \frac{u_3(t)}{m}
\end{aligned} \tag{1.20}$$

Since the satellite is controlled from Earth, then:

$$u(t) = R(x(t), \dot{x}(t)) \tag{1.21}$$

Because of the communication delay (the signal to reach Earth and then back to the satellite) the input to the satellite becomes:

$$u(t) = R(x(t-\tau), \dot{x}(t-\tau)) \tag{1.22}$$

where τ is the transmission time from the satellite to Earth.

The access time delay is the time required by an entity (e.g., a controller, sensor or actuator) to get access to shared media. This situation can be found in a networked control system. The access time delay can be large. If the sensors, actuators and controllers are connected through a network, then the data to the controller is a delayed version of the current states' values. When the controller initiates the control action, (e.g. state feedback) this may also be delayed. Then the system can be described by:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ u(t) &= -Kx(t-\tau) \\ \tau &= \tau_{sc} + \tau_{ca} \end{aligned} \tag{1.23}$$

where τ_{sc} is the delay from the sensor to the controller, and τ_{ca} is the delay from the controller to the actuator.

The above mentioned types of time delay, namely: model approximation, systems nature, transport delay and the communication delay, are the main sources of delay. Most of the time delay systems can be classified into one of these types or a hybrid. In the following section some examples from different fields are given briefly.

1.2.5 More Examples

Network congestion control: In communication networks, a congestion control scheme is usually required. The congestion control schemes are implemented in the networks' nodes (the routers or the switches). To avoid congestion, the amount of data ($x(t)$) in the buffer of a node must be kept below a threshold $\bar{X}$. When $x(t)$ exceeds the threshold $\bar{X}$ the node must act to get $x(t)$ below $\bar{X}$ again. This node receives traffic $Z(t)$ coming from the end nodes (the network's user). The amount of this traffic $Z(t)$ depends on the previous load in the network (some protocols such as TCP, ATM and Frame relay make congestion avoidance schemes based on information about the previous load in the network). This system can be described by the following equations, see Figure 1.6:

$$\begin{aligned} \dot{x}(t) &= z(t-\tau_1) - \mu \\ \dot{z}(t) &= -a(x(t-\tau_2) - \overline{x}) - b(x(t-\tau_2-r) - \overline{x}) \end{aligned} \tag{1.24}$$

where τ_2 is the delay for the nodes status information to reach the end node, and τ_1 is the time for the traffic to come from the end nodes to the node. The time delay is clear in the system equations.

Nuclear reactors [1]: In a nuclear reactor, time is required for heated materials to move through different parts of the system. Currently, the trend is toward using smaller and faster reactors to generate more power. The new reactors need better thermodynamic efficiency to operate at a temperature closer to the upper limit. Then more attention is needed to observe the time delay effect in the reactor [49].

Neural networks: Recently, time delay has been considered in the models of the neural network [1]. Previously, instantaneous propagation of information between the neurons was assumed. Recently, however nonzero propagation time has been observed.

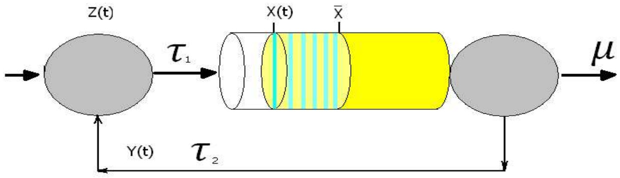

Figure 1.6: Network congestion control

This propagation time adds time delays to the neural network models.

Biology: One example of time delay in biology is the evolution of a single species consuming a common self-renewing food. This process can be described by the following equation:

$$\dot{x}(t) = \gamma[1 - K^{-1}x(t-h))]x(t) \tag{1.25}$$

where h is the time required to produce the food. At any instant t the available amount of food depends on the population of that species at the time instant $t-h$, and this amount affects the rate of change of this species. A similar situation can be used to describe a predator-prey system.

Medicine: In the process of regulating the glucose and insulin in a human body, the pancreas secretes an amount of insulin depending on the glucose concentration in the blood. The pancreatic secretion of insulin at time t is proportional to the value of the glucose at time $t-b$. One of the models for this process is given by:

$$\begin{aligned} \frac{dG}{dt} &= -b_1 G(t) - b_2 I(t) G(t) + b_3 \\ \frac{dI}{dt} &= -b_4 I(t) + \frac{b_6}{b_5} \int_{t-\tau(t)}^{t} G(s) ds \end{aligned}$$

where b_i $i = 1, 2, .., 7$ are constants. This model shows the time delay in this process.

Useful delay: In some cases, the delay may be intentionally introduced into a system to improve its performance. This delay should be introduced carefully in order to obtain the required target. For example, a delay can be used to reduce the overshoot and to yield a smooth and fast transient response [1].

1.3 Summary and Layout

This thesis is concerned with the stability of dynamical systems with time delay. Many traditional methods and tools are available to check the stability for systems without delay, but more work is needed to check the stability of time delay systems. Different methods have been developed to check the stability of time delay systems. These methods are either dealing with special types of delay or giving relatively conservative results. Moreover, designing a controller for time delay systems also remains an open area for research.

In recent years, the time delay systems attracted the interest of many researchers for many reasons that include:

- The progress in the computation capabilities introduces new tools to solve inequality and optimization problems. These tools can be used for time delay systems analysis.

- Future directions in control are more directed toward systems in outer space and in networked control systems [38]. In the first direction, the use of satellites and spacecraft inherently contains some communication delay of the order of seconds. In the other direction, networked control systems contain both access and communication delays. The success in these directions depends on studying the delay effect properly.

Based on this discussion, it becomes clear that:

- time delay exists in many fields;
- effective tools are required to analyze time delay systems; and
- advances in science and technology offers tools which can be use to study time delay systems.

For these reasons a lot of research work has been done recently to study the time delay systems' stability, responses, performance, robustness, etc.

The purpose of this thesis is to study the mathematical tools that can be used to check the stability of time delay systems. Different tools are available for this purpose in both the time domain and the frequency domain. **Chapter 2** explains the characteristics and the developed tools to check the stability of time delay systems. Many directions can be followed to check time delay systems' stability, but the selected one is discussed and justified there. This direction is based on using the Lyapunov-Krasovskii theorem for deterministic, continuous time systems. **Chapter 3** contains a survey of the research made in this direction. Chapter 3 contains also a comparison among the results of recently developed methods. In **Chapter 4**, some theorems are introduced to check the stability of linear systems with varying time delay. These theorems are proved to have better results than earlier methods in terms of the conservatism and complexity. These theorems are extended to design different types of feedback controllers to ensure the system stability while preserving an upper-bound on the $\mathcal{L}_2-$gain of the disturbance. In **Chapter 5**, further simplifications on the methods in Chapter 4 are presented, and an extension is made to cover a set of nonlinear systems. Steps like those made in chapter 4 for the stability, stabilizability and robustness are also made here. In **Chapter 6**, the interval delay type (a delay that has both upper and lower bounds) is presented. Stability and feedback stabilization are studied for this type of system. All the developed methods in Chapter 4, 5 and 6 are verified through simulations. **Chapter 7** contains the conclusions, and it presents some ideas for a possible future extension the thesis work.

Notations and Facts: In the sequel, the following notations and terms are used throughout the thesis: x is the vector of n elements which represents the states of the system; u represents the input vector; y represents the output vector; z represents the controlled output vector; w represents the disturbance. The Euclidean norm is used to represent the magnitude of the vectors. We use W^t and W^{-1} to denote the transpose and the inverse of any square matrix W, respectively. We use $W > 0$ $(\geq, <, \leq 0)$ to denote a symmetric positive definite (positive semi definite, negative, negative semi definite) matrix W, and I to denote the n by n identity matrix. $\Re^+$ and N denote, respectively, the non-negative real numbers and the finite set of integers $1, ..., N$. The symbol $\bullet$ will be used in some matrix expressions to induce a symmetric structure. That is, if given matrices $L = L^t$ and $R = R^t$ of appropriate dimensions, then:

$$\begin{bmatrix} L & N \\ N^t & R \end{bmatrix} = \begin{bmatrix} L & N \\ \bullet & R \end{bmatrix}$$

Sometimes, the arguments of a function will be omitted when no confusion can arise.

CHAPTER 2

TIME DELAY SYSTEMS ANALYSIS

2.1 Terminology and Basic Concepts

Chapter 1 shows the importance of considering time delay effect in some systems. This section explains some of the terminologies used in the analysis of time delay systems.

Consider the following functional differential equation:

$$\dot{x}(t) = a_0 x(t) + a_1 x(t-\tau) + u(t) \tag{2.1}$$

Here $\dot{x}(t)$ is a function of $x(t)$, $x(t-\tau)$ and the input $u(t)$. To solve $x(t)$ we need $x(t_0)$ as initial conditions in the interval $t - \tau < t_0 < t$. These initial conditions can be defined as:

$$x_t = x(t) = \phi(t), \quad t \in [-\tau, 0], \tag{2.2}$$

where $\phi : [-\tau, 0] \rightarrow \Re^n$. Therefore, $\dot{x}(t)$ is a function of $\phi(t)$.

In general, a retarded functional differential equation can be described by:

$$\dot{x}(t) = f(t, x_t, u) \tag{2.3}$$

where $f(., ., .)$ can be a nonlinear time varying functional that takes real value t, a function x_t and input u to produce a vector of n real numbers. If $f(., ., .)$ is not a function of $\dot{x}(t-\tau)$, then it is called a retarded functional. On the other hand, if f is a function of $\dot{x}(t-\tau)$ (which can be found in some systems), then it is called neutral functional. The general form of neutral time delay systems is given by:

$$\dot{x}(t) = f(t, x_t, \dot{x}_t, u) \tag{2.4}$$

There is a lot of research on this type of systems, but it is beyond the scope of this thesis.

If $f(., ., .)$ is linear, then the Eqn 2.3 becomes:

$$\dot{x}(t) = A(t)x_t + u(t) \tag{2.5}$$

According to [15] it is always possible to find a matrix function $F : \Re^n[-\tau, 0] \rightarrow \Re^{n,n}$ of a bounded variation such that $F(t, 0) = 0$ and:

$$A(t)x_t = \int_{-\tau}^{0} (d_\theta[F(t,\theta)]x(\theta)) \tag{2.6}$$

By using Eqn 2.6, Eqn 2.5 becomes:

$$\dot{x}(t) = \int_{-\tau}^{0} (d_\theta[F(t,\theta)]x(\theta)) + u(t) \tag{2.7}$$

The system in 2.7 is called a system with distributed delays. Chapter 3 shows that a system with distributed delays can be approximated or transformed into a system with multiple discrete delays.

When the system is time invariant, Eqn 2.5 becomes:

$$\dot{x}(t) = Ax_t + u(t) \tag{2.8}$$

Here the response at $t = t_1$ for given initial conditions at $t = t_0$ depends only on the value of $t_1 - t_0$, not on the individual values of t_1 and t_0.

Generally, differential equations with time delay lead to distributed systems which have an infinite number of states [15] [13]. The conventional methods to analyze systems in terms of the controllability, observability, stability etc., should be revised to deal with time delay systems [13].

2.2 Stabiliy of Time Delay Systems

Systems with time delay have attracted the interest of many researchers since the early 1900s. During that time, delay was included in many models of systems in different fields, like those described in Chapter 1. In the 1940s, some theorems were developed to check the stability of time delay systems in the frequency domain. The corresponding theorems in the time domain appeared in the 1950s and 1960s. In the last 20 years, the improvement in the computation tools gave an opportunity to develop new methods to check the stability of time delay systems.

The available tools to check the stability of time delay systems can be classified into two categories: delay-independent methods or delay-dependent methods. These two categories are covered in the following subsections.

2.2.1 Delay-Independent Stability Methods

Delay-independent stability methods check whether the stability of a time delay system is preserved for a delay of any size or not. The methods in this category try to check if the magnitude of the delayed states does not affect the stability of the system , no matter what the value of that delay is. These methods are easier to derive, but they suffer some conservatism because:

- not all the systems have insignificant delayed states;
- in many cases the delay is fixed, and so applying these methods imposes unnecessary conditions;
- in most of the systems, the delay has a relatively small upper bound, even if it is not fixed;
- delay-independent stability methods can be used only when the delay has a destabilizing effect.

For these reasons and others, many researchers have shifted their interests to the delay-dependent stability methods.

2.2.2 Delay-Dependent Stability Methods

In contrast to delay-independent stability methods, delay-dependent stability methods require some information about the delay. This information serves one of the following two purposes:

- to ckeck whether a given system, with some dynamics and delay information, is stable or not; or

- to check for how long delays a given a system, with some dynamics, can preserve its stability.

Generally, the second purpose is used to qualify any developed method. For implementation purposes, the conditions for time delay systems can only be sufficient. Different methods give different sets of conditions. It is necessary to know for how much delay each method can prove the stability of the system. In a set of methods, the best one must prove the stability of the system for the largest delay. After that, this method can be used to check the stability of a specific system which has a specific delay (the first purpose). In research, the commonly used delay types are:

1. fixed delay

   $\tau = \rho, \quad \rho = constant$

2. unknown time-varying delay with an upper-bound

   $0 \leq \tau \leq \rho, \quad \rho = constant$

3. unknown time-varying delay with an upper-bound on its value and an upper-bound on its rate of change

   $0 \leq \tau \leq \rho, \quad \rho = constant$

   $\dot{\tau} \leq \mu$

4. delay that varies within some interval (interval delay type)

   $h_1 \leq \tau \leq h_2$, $h1,\ h2$ are constants

5. delay that varies within some interval with an upper-bound on its rate of change

   $h_1 \leq \tau \leq h_2$, $h1,\ h2$ are constants

   $\dot{\tau} \leq \mu$

2.3 Time Delay Stability Methods in Frequency and Time Domains

Generally, the available tools to check the stability of time delay system are either in the frequency domain or in the time domain. An overview of the methods in each domain is given in this section.

2.3.1 Frequency Domain Methods

As in systems without delay, the frequency domain methods can be used only with Linear Time Invariant (LTI) systems. Consequently, they can be used only for delay-independent category or fixed delay type. One research direction in the frequency domain tries to find the roots of the characteristic equation of the time delay systems. For example, consider the following system:

$$\dot{x}(t) = Ax(t) + \sum_{k=1}^{K} A_k x(t - \tau_k) \tag{2.9}$$

The characteristic equation for this system is given by:

$$p(s; e^{-r_1 s}; e^{-r_2 s}; ...; e^{-r_K s}) = det(sI - A - \sum_{k=1}^{K} A_k e^{-r_k s}) \tag{2.10}$$

According to [15], if the system in Eqn 2.9 is stable for τ_k =0, $k = 1, 2.., K$, then it is stable for small $\tau_k s > 0$. The values of $\tau_k > 0$ can increase continuously without losing the stability up to some values. As τ_k increases, one or more of the roots of Eqn 2.10 may fall in the right half plane (RHP) of the S domain which implies instability. The approach in this direction is to find ways to track the roots of the characteristic equation as τ_k increases. It is necessary to find the values of τ_k that make, at least, one of the roots of Eqn 2.10 fall in the RHP. If the characteristic equation of a system has no root in the RHP for all τ_k, then the system is delay-independently stable.

Another direction in the frequency domain tries to approximate the term $e^{-r_k s}$ (the representation of delay in the S domain) by a rational polynomial $G_r(s, r_k)$. Many approximations were developed on different criteria. Al-Amer and Al-Sunni in [41] developed an approximation to keep the H_∞ norm of $|G_r(s, r_k) - e^{-r_k s}|$ below a specific value. By this approximation, the transfer function of a time delay system is transformed into another one without time delay. Hence, the ordinary tools of the frequency domain can be used.

The results in the frequency domain are good and quite acceptable. For this reason, and because it deals only with LTI systems with fixed delay, the frequnecy domain approach is not considered in this thesis.

2.3.2 Time Delay Stability Methods in Time domain

Time domain methods can be used for general systems, not necessarily LTI systems. They also can tackle different delay types. In the time domain, Lyapunov's theorem can be used to check the stability of the system. One difference between constructing Lyapunov functions for systems with and without delay is the dependence of time delay systems on their previous states' values. It is expected that the selected Lyapunov function should have terms to consider these delayed states.

Based on Lyapunov's theorem, there are two main theorems to check the stability of time delay systems: the Lyapunov-Razumikhin theorem and the Lyapunov-Krasovskii theorem.

Lyapunov-Razumikhin Theorem

Because the evolution of the states in time delay systems depends on the current and previous states' values, their Lyapunov functions should become functionals (more details in Lyapunov-Krasovskii method). The functional may complicate the formulation of the conditions and their analysis. To avoid such complications, Razumikhin developed a theorem which will construct Lyapunov functions but not as functionals. To apply the Razumikhin theorem, one should build a Lyapunov function $V(x(t))$. This $V(x(t))$ is equal to zero when $x(t) = 0$ and positive otherwise. The theorem does not require $\dot{V}$ to be less than zero always, but only when $V(x(t))$ becomes greater than or equal to a threshold $\bar{V}$. $\bar{V}$ is given by:

$$\bar{V} =^{max}_{\theta \in [-\tau, 0]} \; V(x(t+\theta)) \tag{2.11}$$

Based on this condition, one can understand the theorem statement, which is ([15]):

Theorem 2.1 *Suppose f is a functional that takes time t and initial values x_t and gives a vector of n states $\dot{x}$, and u, v and w are class $\mathcal{K}$ functions $u(s)$ and $v(s)$ are positive for $s > 0$ and $u(0) = v(0) = 0$, v is strictly increasing. If there exists a continuously differentiable function $V : \mathcal{R} \; X \; \mathcal{R}^n \rightarrow \mathcal{R}$ such that:*

$$u(\|x\|) \leq V(t, x) \leq v(\|x\|) \tag{2.12}$$

and the time derivative of $V(x(t))$ along the solution $x(t)$ satisfies $\dot{V}(t, x) \leq -w(\|x\|)$ whenever $\bar{V} = V(t+\theta, x(t+\theta)) \leq V(t, x(t))$, $\theta \in [-\tau, 0]$, then the system is uniformly stable. If in addition $w(s) > 0$ for $s > 0$ and there exists a continuous non-decreasing function $p(s) > s$ for $s > 0$ such that $\dot{V}(t, x) \leq -w(\|x\|)$ whenever $V(t+\theta, x(t+\theta)) \leq p(V(t, x(t)))$ for $\theta \in [-\tau, 0]$, then the system is uniformly asymptotically stable. If in *addition lim* $u(s)_{s\to\infty} = \infty$ *then the system is globally asymptotically stable.*

Here $\bar{V}$ serves as a measure for $V(x(t))$ in the interval from $t-\tau$ to t. If $V(x(t))$ is less than $\bar{V}$, $\dot{V}$ could be greater than zero. On the other hand, if $V(x(t))$ becomes greater than or equal to $\bar{V}$, then $\dot{V}$ must be less than zero, such that V will not grow beyond limits. In other words, according to the Razumikhin theorem, $\dot{V}$ need not be always less than zero, but the following conditions should be satisfied:

$$\dot{V} + a(V(x) - \bar{V}) \leq 0 \tag{2.13}$$

for $a > 0$. Therefore, there are three cases for the system to be stable:

1. $\dot{V} < 0$ and $V(x(t)) \geq \bar{V}$. Here the states do not grow in magnitude;
2. $\dot{V} > 0$ but $V(x(t)) < \bar{V}$. In this case, although $\dot{V}$ is positive (the values of the states increase), the Lyapunov function is limited by an upper bound; and
3. a case where both terms are negative.

The condition in 2.13 ensures uniform stability, i.e. the states may not reach the origin, but they are contained in some domain. To ensure the asymptotic stability, the condition should be:

$$\dot{V} + a(p(V(x(t))) - \bar{V}) < 0,\ a > 0 \tag{2.14}$$

where $p(.)$ is a function with the property: $p(s) > s$.

This condition implies that when the system reaches some value which makes $p(V(x(t))) = \bar{V}$, then $\dot{V}$ should be negative and $V(x(t))$ will not reach $\bar{V}$. In the coming interval τ, $V(x)$ will never reach the old $\bar{V}$ ($\bar{V}_{old}$). The maximum value of V

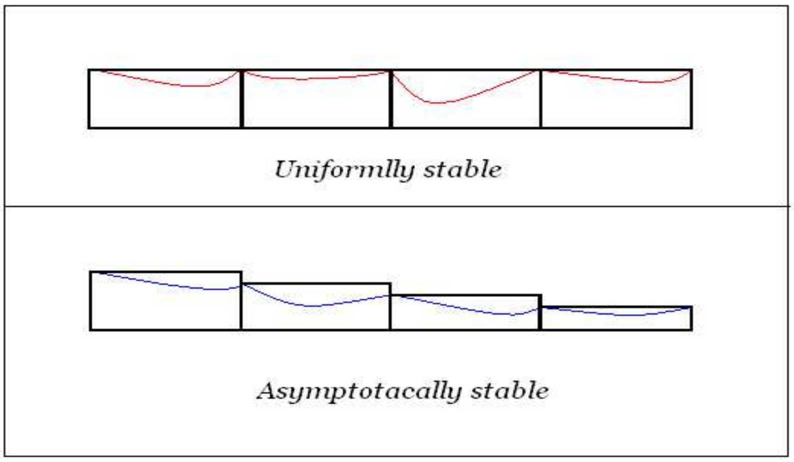

Figure 2.1: Razumakhin method

in this interval is the new $\bar{V}$ ($\bar{V}_{new}$) which is less $\bar{V}_{old}$. With the passage of time, V keeps decreasing until the states reach the origin (see Figure 2.1).

Lyapunov-Krasovskii Theorem

While Razumikhin's theorem is based on constructing Lyapunov functions, the Lyapunov-Krasovskii theorem constructs functionals instead. Based on the Lyapunov theorem's concept, the function V is a measure of the system's internal energy. In time delay systems, the internal energy depends on the value of x_t, and it is reasonable to construct V which is a function of x_t (which is also a function). Because V is a function of another function, it becomes a functional. To ensure asymptotic stability, $\dot{V}$ should always be less than zero. The Lyapunov-Krasovskii theorem is discussed in more detail in Section 2.4.

From the results obtained in the recent research, most of the methods based on Lyapunov-Razumikhin are found to be special cases of corresponding methods based on Lyapunov-Krasovskii. This means that the former is more conservative [15]. For this reason, most of the recent research (and the present thesis as well) are based on the Lyapunov-Krasovskii theorem.

2.4 Lyapunov-Krasovskii Theorem

Previously, methods based on Lyapunov-Krasovskii were criticized for being applicable only to a subset of the delay types mentioned in Section 2.2. Furthermore, to be included in methods based on Lyapunov-Krasovskii, the delay rate of change must be ≤ 1 [13]. Recent results succeeded in resolving these problems (more details in Chapter 3). The remaining advantage of Razumikhin-based methods over Krasovskii is their relative simplicity, but Lyapunov-Krasovskii gives less conservative results. Before discussing the theorem, we have to define the following notations:

$$\begin{aligned} \phi &= x_t \\ \|\phi\|_c &= \max_{\theta\in[-\tau,0]} x(t+\theta) \end{aligned} \tag{2.15}$$

The statement of the Lyapunov-Krasovskii theorem given in ([15]) is:

Theorem 2.2 *Suppose f is a functional that takes time t and initial values x_t and gives a vector of n states $\dot{x}$. u, v and w are class $\mathcal{K}$ functions $u(s)$ and $v(s)$ are positive for $s > 0$ and $u(0) = v(0) = 0$, v is strictly increasing. If there exists a continuously differentiable function V such that:*

$$u(\|\phi\|) \leq V(t, x_t) \leq v(\|\phi\|_c) \tag{2.16}$$

and the time derivative of V along the solution $x(t)$ satisfies $\dot{V}(t, x_t) \leq -w(\|\phi\|)$ for $\theta \in [-\tau, 0]$ then the system is uniformly stable. If in addition $w(s) > 0$ for $s > 0$ then *the system is uniformly asymptotically stable. If in addition lim* $u(s)_{s\to\infty} = \infty$ *then the system is globally asymptotically stable.*

It is clear that V is a functional and $\dot{V}$ must always be negative.

A theorem based on Lyapunov-Krasovskii was developed in [46] for a simple linear time invariant system with multiple discrete time delays. The system is given by:

$$\dot{x}(t) = A_0 x(t) + \sum_{j=1}^{m} A_j x(t - h_j) \tag{2.17}$$

where $h_j,\ j = 1, 2..., m$ are constants. The Lyapunov-Krasovskii functional that gives a necessary and sufficient condition for the stability of this system is:

$$\begin{aligned} V(x_t) &= x'(t)U(0)x(t) \\ &+ \sum_{k=1}^{m}\sum_{k=1}^{m} x'(t+\theta_2)A'_k \int_0^{-h_k} U(\theta_1 + \theta_2 + h_k - h_j)A_j x(t+\theta_1)d\theta_1 d\theta_2 \\ &+ \sum_{m}^{k=1} \int_0^{-h_k} x'(t+\theta)[(h_k+\theta)R_k + W_k]x(t+\theta)d\theta \end{aligned} \tag{2.18}$$

where $W_0, W_1, ..., W_m, R_1, R_2, ...$ and R_m are positive definite matrices and U is to be obtained from the following equation:

$$\frac{d}{d\tau}U(\tau) = U(\tau)A_0 + \sum_{k=1}^{m} U(\tau - h_k)A_k \qquad \tau \in [0, max_k(h_k)] \tag{2.19}$$

The functional in Eqn 2.18 is formulated by imitating the case of delay-free systems. Therein, the state transition matrix is found and then used to find $P > 0$ that makes:

$$x'(t)(PA + A'P)x(t) = -Q \quad Q > 0$$

The Lyapunov functional in Eqn 2.18 gives the necessary and sufficient conditions for the system stability; but finding the U for this equation is very difficult. It involves solving algebraic ordinary and partial differential equations with appropriate boundary conditions, which is obviously unpromising [15]. Even if we can find this U, the resulting functional will lead to a complicated system of partial differential equations yielding infinite dimensional LMI. For this reason, many researchers consider simplified forms of Eqn 2.18. The simplified forms introduce simpler but more conservative, sufficient conditions. These sufficient conditions can be represented by an appropriate set of linear matrix inequalities (LMIs).

The Lyapunov functional in Eqn 2.18 was derived for an LTI system with fixed time delays. For time varying delay and/or nonlinear systems, the Lyapunov functional becomes more complicated. However, looking at the terms in Eqn 2.18, one can surmise the possible terms which can be included in the simplified functionals.

The present thesis adopts a simplified Lyapunov-Krasovskii functional that gives sufficient conditions for the stability of a system with a single time varying delay. Obviously, to be more general, one should consider nonlinear systems with distributed delays. It is well known that dealing with nonlinear systems may not give results that are general enough, because every family of nonlinear systems has its own characteristics. Furthermore, dealing with nonlinear systems is very difficult, even for systems without time delay. The general practice is to make a linearization around some operating point. This linearized model can be analyzed while treating the nonlinearities as perturbations. However, the method proposed in Chapter 5 can be used for some families of nonlinear systems which are not necessarily coming from a linearized mode. Distributed delay is also difficult to deal with, and many systems have a discrete type of delay. In addition, there are possibilities to approximate [13] or transform [12] the distributed delay system into a system with multiple discrete delays. Chapter 3 shows that, if the Lyapunov functional is selected properly, a method developed for single delay can easily be extended to cover the cases wherein we have multiple delays. In the present thesis, time varying delay is considered, because it covers large class of systems and it can be modified to cover fixed delay. Delay-independent methods are avoided to get more general results, since any system which satisfies delay-independent stability conditions will also satisfy the delay-dependent stability conditions for any value of the delay.

As a conclusion of the section, this present thesis will use the Lyapunov-Krasovskii theorem to check the delay-dependent stability of an uncertain continuous-time linear and time-invariant system with time varying delay. The next chapter contains a survey of the research done in this area.

CHAPTER 3

LITERATURE SURVEY

The previous two chapters shows the importance of considering the effect of time delay on systems. Amongst the available tools to check the stability of time delay systems, the Lyapunov-Krasovskii method is found to be one of the most efficient. This chapter contains a literature survey for this method.

3.1 Historical Review

The first step to develop a method based on Lyapunov-Krasovskii is to build an appropriate Lyapunov functional. This functional should be selected carefully so that the resulting conditions (which are sufficient conditions) can easily be checked. The selected functional should yield the least conservative results possible. After selecting the Lyapunov functional, one can improve the result by many techniques such as: introducing free matrices, making some bounding, using algebraic inequalities, etc.

When studying or developing a method based on Lyapunov-Krasovskii, important points to be considered may include:

1. the selected delay type;

2. the selected Lyapunov-Krasovskii functional;

3. the considered type of uncertainties;
4. the number of free matrices introduced if any;
5. the bounding inequality employed if any; and
6. the stabilization possibility.

These points give an indication about the strength and complication of the method under consideration. The last point is very important, since it adds value to the method. For a method which has the stabilization possibility, it may design a controller to ensure the stability.

In early Lyapunov-Krasovskii based research, model transformations and upper-boundings were essential parts in the conditions' derivation. For example, the following system is treated in [15]:

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau) \tag{3.1}$$

When the following transformation is used:

$$x(t-\tau) = x(t) - \int_{t-\tau}^{t} \dot{x}(s)ds$$

With this transformaion Eqn 3.1 became:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d[x(t) - \int_{t-\tau}^{t} \dot{x}(s)ds] \\ &= (A + A_d)x(t) - A_d \int_{t-\tau}^{t} \dot{x}(s)ds \end{aligned} \tag{3.2}$$

To check the stability, the considered Lyapunov functional was:

$$V(x_t) = V_1 + V_2 + V_3$$

where

$$\begin{aligned} V_1 &= x'(t)Px(t), \\ V_2 &= \int_{t-\tau}^{t} x'(s)Qx(s)ds \\ V_3 &= \int_{-\rho}^{0}\int_{t+\theta}^{t} \dot{x}'(s)A_d'XA_d\dot{x}(s)ds \end{aligned} \tag{3.3}$$

The expression of $\dot{V}(x_t)$ included the term:

$$-2x'(t)PA_d\int_{t-\tau}^{t} \dot{x}(s)ds \tag{3.4}$$

This term is neither positive nor negative definite. With this term, one cannot prove whether $\dot{V}(x_t)$ is negative or not. There was a common practice to resolve this term by replacing it with an upper-bound. This upper-bound should be a summation of only positive and negative definite terms. In [15] the following inequality was used:

$$-2a'b < a'Xa + b'X^{-1}b \quad X > 0 \tag{3.5}$$

Upper bounding helps in solving some problems, but it adds conservatism to the method. Since positive terms are added to $\dot{V}(x_t)$, therefore $\dot{V}(x_t)$ has less chance to become negative. To reduce the conservatism, X should be selected to give the smallest possible upper-bound, i.e. replacing $-2a'b$ with M which is given by:

$$M =_{X>0}^{inf} (a'Xa + b'X^{-1}b) \tag{3.6}$$

In this typical example, two practices were used during the derivation:

1. *model transformation*: which is the substitution of $x(t-\tau)$ by $x(t) - \int_{t-\tau}^{t} \dot{x}(s)ds$. In early research the transformation was used to treat delayed feedback. Hence, $A_d = B * K$. The practice was to assume zero time delay, such that the closed loop A_c becomes $A + A_d$. The delayed terms were considered as disturbance. The system in Eqn 3.2 can be considered as:

$$\dot{x}(t) = A_c x(t) + O_T \tag{3.7}$$

   where $O_T \rightarrow 0$ as $\tau \rightarrow 0$. For a very small delay O_T becomes:

$$O_T = \int_{t-\tau}^{t} \dot{x}(s)ds \approx \tau \dot{x}(t) \approx 0 \tag{3.8}$$

   By such a transformation, delay-dependent category was introduced in time delay systems' analysis. The size of the delay affects the stability of the system, and then the system is delay-dependent stable. Model transformation is a source of conservatism in any developed method to check the stability of time delay systems [16] [17]. It introduces additional dynamics to the system. As τ increases, the added dynamics may become unstable before the dynamics of the original system [15].

2. *bounding of some terms*: which adds positive values to $\dot{V}$. The bounding may have a large effect in the conservatism of the methods.

3.2 Literature Review

This section reviews the Lyapunov-Krasovskii based methods. The extracted remarks from these methods were very helpful in setting research guidelines for the present thesis.

Park in [35] showed the types of delay that were used in previous research. These types were: 1) unbounded (for delay independent criteria) 2) fixed known delay (can be solved by Smith predictor), and 3) fixed unknown delay type. In his work, Park focused on the third type. He showed that the inequality in 3.5 is very conservative. To reduce this, Park suggested the following inequality instead:

$$-2a'b < (a + Mb)'X(a + Mb) + b'X^{-1}b + 2b'Mb \quad X > 0 \tag{3.9}$$

Here M can take any value. When $M = 0$, the inequality in 3.9 reduces to the inequality in 3.5. So 3.5 is a special case of 3.9. In its worst case, the inequality in 3.9 can give the same result as the one in 3.5 by setting $M = 0$. Park found less conservative results than the earlier work.

A summary of the method in [35]:

- it uses first-order transformation given in Eqn 3.2;
- it proposes new bounding technique;
- it considers unknown fixed delay;
- it uses 3 Lyapunov terms;
- it uses 3 parameter matrices;
- it uses 2 free-weighting matrices; and

- it does not consider any sort of uncertainty.

Moon and Park in [60] used the unknown fixed delay type. They made further improvements over the inequality in 3.5. The conservatism was reduced by using the following inequality:

$$
\begin{aligned}
-2\int a'(s)Nb(s)ds \;&\leq\; \int \begin{bmatrix} a(s) \\ b(s) \end{bmatrix}' \begin{bmatrix} X & Y-N' \\ \bullet & Z \end{bmatrix} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix} ds \\
\begin{bmatrix} X & Y \\ \bullet & Z \end{bmatrix} \;&>\; 0
\end{aligned}
\tag{3.10}
$$

With suitable substitution for Z and Y, one can show that inequalities in 3.5 and 3.9 are special cases of 3.10. Thus, it may give less conservative results.

Moon and Park extended their method to design a memory-less state feedback controller. During the formulation of the stabilization theorem, nonlinear matrix inequalities resulted. These included three parameter matrices S, L and R and their inverses T, J and K. These matrices are to be selected such that the stability and stabilization conditions are satisfied. Moon and Park solved the problem by using an iterative method that works as follows:

- first, the system has to be stable for some fixed delay ρ;
- by using this ρ and from stability LMIs, find the values of S, L and R;
- for the same ρ, consider S, L and R as constants in the stabilization LMIs;
- solve the stabilization LMIs while minimizing the sum of the products of the old values of S, L and R matrices and T, J and K given by the stabilization LMI;

- every time, check some conditions relating to the matrices' values. If the conditions are satisfied, a stabilizing controller can be designed for this ρ;
- if not, use the inverse of the T, J and K as new values for S, L and R, and run the stabilization LMI again until the condition is satisfied; and
- increase ρ by a small value, and use the last values for the S, L and R matrices as constant, and repeat the process.

One may notice some important points about this iterative method:

- the method should start with a stable system for some ρ, and then the method cannot be used with unstable systems; and
- it follows an iterative algorithm, and then the size of the LMIs is relatively large, and the algorithm may take a long time to give the result.

The possibility to extend any method for designing a stabilizing controller, as discussed previously, is a very important criterion in evaluating the method. Many methods try to get better stability results by adding more terms to the Lyapunov functional which complicate the conditions, and hence stabilizing controller design.

A summary of the method in [60]:

- it uses a first-order transformation given in Eqn 3.2;
- it proposes new bounding technique;
- it considers unknown fixed delay;
- it uses 3 Lyapunov terms;
- it uses 3 parameter matrices;

- it uses 2 free-weighting matrices; and
- it considers norm-bounded uncertainties.

Fridman et al. in [3] discussed the sources of conservatism in prior research. They showed that model transformation and bounding of the cross terms mentioned above are the main conservatism sources. The method proposed in [60] was used for bounding the cross terms. Furthermore, Fridman et al. adopted a transformation called descriptor model transformation which is given by:

$$\begin{aligned} \dot{x}(t) &= y(t) \\ 0 &= -y(t) + (A + A_d)x(t) - A_d \int_{t-\tau}^{t} y(s)ds \end{aligned} \tag{3.11}$$

The descriptor model transformation was found to give the least conservative results of all the other transformations. Fridman et al. considered a system with two time delays of the type $0 \leq \tau \leq \rho$, $\dot{\tau} \leq \mu < 1$. Two types of uncertainties were considered: norm-bounded and polytypic uncertainties. The ensuing results were found to be better than those obtained by the method in [60] while using a more general delay type.

A summary of the method in [3]:

- it uses descriptor model transformation;
- it uses bounding proposed by [60];
- it uses delay of the type
  $0 \leq \tau \leq \rho$, $\dot{\tau} \leq \mu < 1$;
- it uses 3 Lyapunov terms;
- it uses 5 parameter matrices;

- it uses 5 free-weighting matrices; and
- it considers norm bounded and polytypic uncertainties.

Min et al. in [32] described the main techniques used by previous researchers to check the stability of time delay systems. They showed that all previous methods used some sorts of transformation and bounding which are sources of conservatism. Then, by avoiding these two practices, one can obtain less conservative results. In the paper [32], Min et al. introduced a new method that contains neither model transformation nor upper-bounding for the cross term. The same delay type assumed in [3] is considered in the paper [32]. The main contribution was the introduction of what was called the *free-weighting matrices* method. This method is based on adding the following zero value term to $\dot{V}$:

$$2[x'(t)Y + x'(t-\tau)T][x(t) - x(t-\tau) - \int_{t-\tau}^{t} x(s)ds]$$

where Y and T are free matrices. These matrices add more freedom to find a valid solution. These matrices introduce the relations between different variables to the Lyapunov functional. Since more information is exploited, better results are expected to be obtained. The selected Lyapunov functional was:

$$\begin{aligned} V &= V_1 + V_2 + V_3 \\ V_1 &= x'(t)Px(t) \quad P > 0 \\ V_2 &= \int_{t-\tau}^{t} x'(s)Qx(s)ds \quad Q > 0 \\ V_3 &= \int_{-\rho}^{0}\int_{t+\theta}^{t} x'(s)Zx(s)dsd\theta \quad Z > 0 \end{aligned} \tag{3.12}$$

$\dot{V}$ is given by:

$$\begin{aligned}
\dot{V} &= \dot{V}_1 + \dot{V}_2 + \dot{V}_3 \\
\dot{V}_1 &= x'(t)(PA + A'P)x(t) + 2x'(t)PA_d x(t-\tau) \\
\dot{V}_2 &= x'(t)Qx(t) - (1-\dot{\tau})x'(t-\tau)Qx(t-\tau) \\
&\leq x'(t)Qx(t) - (1-\dot{\mu})x'(t-\tau)Qx(t-\tau) \\
\dot{V}_3 &= \rho\dot{x}'(t)Z\dot{x}(t) - \int_{t-\rho}^{t} \dot{x}'(s)Z\dot{x}(s)ds
\end{aligned} \tag{3.13}$$

Min et al. considered $x(t)$, $x(t-\tau)$ and $\dot{x}(s)$ as states in the LMIs. Free-weighing matrices were added to separate the problems into two LMIs. More free matrices were introduced to reduce the conservatism. By using a common example, Min et al. showed the superiority of their method over [60] and [3].

A summary of the method in [32]:

- it does not include any model transformation;
- it does not make any bounding;
- it uses delay of the type $0 \leq \tau \leq \rho$, $\dot{\tau} \leq \mu < 1$;
- it uses 3 Lyapunov terms ;
- it uses 3 parameter matrices;
- it uses 6 free-weighting matrices; and
- it considers norm bounded uncertainties.

He et al. in [56] used a method that is similar to the one in [32], but they did not open the term $\dot{x}(t)$ while calculating $\dot{V}$. The resulting LMI does not contain any system matrices. The system matrices were inserted in the LMI through free-weighting matrices. There where six free matrices. Three were inserted through:

$$2[x'(t)N_1 + x'(t-\tau)N_2 + \dot{x}'(t)N_3][x(t) - x(t-\tau) - \int_{t-\tau}^{t} \dot{x}(s)ds]$$

and the other three were inserted using:

$$2[x'(t)T_1 + x'(t-\tau)T_2 + \dot{x}'(t)T_3][-\dot{x}(t) + Ax(t) + A_d x(t-\tau)] \tag{3.14}$$

This became a new way to look at the problem, where the original LMIs do not satisfy the conditions but the introduced free matrices try to make it satisfy them.

A summary of the method in [56]:

- it does not include any model transformation;
- it does not make any bounding;
- it uses delay of the type $0 \leq \tau \leq \rho$, $\dot{\tau} \leq \mu < 1$;
- it uses 3 Lyapunov terms;
- it uses 3 parameter matrices;
- it uses 6 free-weighting matrices; and
- it considers polytypic uncertainties.

Jing et al. in [51] proposed a method for the following delay types:

$$\begin{aligned} &A1) \quad 0 \leq \tau \leq \rho \;\; \dot{\tau} < \mu, \\ &A2) \quad 0 \leq \tau \leq \rho \end{aligned} \tag{3.15}$$

Each of the two types deal with the delay rate of change differently. $A1$ puts some upper-bound in the delay rate of change while $A2$ does not. A method based on the $A2$ type is applicable for a system with any delay rate of change. The supporters of $A1$ consider it as a method for fast dynamics. On the other hand, previously, when a Lyapunov-Krasovskii was being used for $A1$ delay type, it was used only when $\mu < 1$, and it was applicable only for slow dynamics. The discussion in Chapter 2 about delay-independent and delay-dependent methods is applicable here. One can call them the delay rate of change-independent method and the delay rate of change-dependent methods.

Jing et al. made a great contribution by eliminating the need for μ to be < 1. Hence the delay rate of change can take any value. Another important point is the use of augmented terms in the Lyapunov function such as $[x^t(t)\ x^t(t-\tau)]^t$. By using a common example, Jing et al. showed the advantage of their method over [3]

A summary of the method in [51]:

- it does not include any model transformation;
- it does not make any bounding;
- it uses delay of the type; A1) $0 \leq \tau \leq \rho \quad \dot{\tau} < \mu$ A2) $0 \leq \tau \leq \rho$;
- it uses 4 Lyapunov terms;
- it uses 8 parameter matrices;
- it uses no free-weighting matrices; and
- it considers norm-bounded uncertainties.

Xu et al. in [42] used the same method proposed in [32] to develop a theorem for systems with fixed unknown delay. The paper [42] is mentioned here for two purposes. First, it shows that one can develop theorems for the delay of the type $0 \leq \tau \leq, \rho \quad \dot{\tau} \leq \mu$, then these theorems can be manipulated to deal with fixed delay systems. Second, by comparing the results obtained by Moon et al. in [60] and those obtained by Xu et al. the advantage of the latter over the former becomes clear. The LMIs of Moon et al. are:

$$\begin{aligned}
&\begin{bmatrix} PA + A'P + \rho X + Y + Y' + Q & -Y + PA_d & \rho A'Z \\ \bullet & -Q & \rho A_d'Z \\ \bullet & \bullet & -\rho Z \end{bmatrix} < 0 \\
&\begin{bmatrix} X & Y \\ \bullet & Z \end{bmatrix} > 0
\end{aligned} \tag{3.16}$$

while the one in developed by Xu et al. is:

$$\begin{bmatrix} PA + A'P + Y + Y' + Q & -Y + PA_d & -\rho Y & \rho A'Z \\ \bullet & -Q - W - W' & \rho W & \rho A_d'Z \\ \bullet & \bullet & -\rho Z & 0 \\ \bullet & \bullet & \bullet & -\rho Z \end{bmatrix} < 0 \tag{3.17}$$

where W is a free matrix. By assuming $W = 0$ and by Schur's complement, Eqn 3.17 can be written as:

$$\begin{bmatrix} PA + A'P + Y + Y' + Q + \rho Y Z^{-1} Y' & -Y + PA_d & \rho A'Z \\ \bullet & -Q & \rho A_d'Z \\ \bullet & \bullet & -\rho Z \end{bmatrix} < 0 \tag{3.18}$$

From Eqn 3.16 and by using Schur's complement, it is clear that $\rho X > \rho Y Z^{-1} Y'$. Then Eqn 3.16 is more conservative than Eqn 3.17. This is obtained from a special case in Eqn 3.17. Since W may be selected to make Eqn 3.17 more negative, then the result is expected to be less conservative. In conclusion, using the free-weighting matrices was proved mathematically to be less conservative than various bounding and model transformation based methods.

A summary of the method in [42]:

- it does not include any model transformation;
- it does not make any bounding;
- it considers unknown fixed delay;
- it uses 3 Lyapunov terms;
- it uses 3 parameter matrices;
- it uses 2 free-weighting matrices; and
- it considers no uncertainties.

Lin et al. in [36] made a Lyapunov functional that contains augmented terms similar to those in [42]. In the paper [36], Lin et al. did not open the term $\dot{x}(t)$. They introduced the system matrices with free-weighting matrices. Nine free matrices were used. In doing this the work in [42] is combined with the work in [56].

A summary of the method in [36]:

- it does not include any model transformation;
- it does not make any bounding;
- uses delay of the type; $0 \leq \tau \leq \rho$, $\dot{\tau} \leq \mu < 1$;
- it uses 3 Lyapunov terms;

- it uses 7 parameter matrices;
- it uses 9 free-weighting matrices; and
- it considers polytypic uncertainties.

He et al. in [54] used the ideas introduced in [56] and [42] to build a method for systems with multiple fixed delays. They showed that a well formulated Lyapunov-Krasovskii functional for a system with single delay can easily be extended to treat a system with multiple discrete delays. The case of two time delays was studied, showing that any number of time delays can be included easily. He et al. succeeded in making the single delay case as a special case of their developed method, i.e. if all the time delays are given the same value, the resulting LMI is one of the theorems developed previously for a single time delay, while this is not the case in previous work e.g. [3]. The paper [54] is mentioned here to justify the selected direction of dealing with single time delay. One can concentrate on the simplest case (single time delay) and try to find the best possible results there. Then, this method can be expanded for multiple discrete delays which can be used to cover systems with a distributed time delay (see Chapter 2).

A summary of the method in [54]:

- it does not include any model transformation;
- it does not make any bounding;
- it considers multiple fixed delays;
- it uses 6 Lyapunov terms;
- it uses 6 parameter matrices;

- it uses 36 free-weighting matrices; and
- it considers no uncertainties.

Park et al. in [34] used the Lyapunov functional used in [32], but they added the following term:

$$V_4 = \int_{t-\rho}^{t} x'(s)Q_2x(s)ds \tag{3.19}$$

The formulated LMI contains terms to check the H_∞ gain from the disturbances to a controlled output. Park et al. prsesented a new method to formulate the inequalities in the paper. The new method is based on using matrices e_i i=1, 2, 3, 4 and 5. By using e_i, Park et al. avoided using any Schur's complement during the derivation. The obtained results are found to be better than those in [32]. One disadvantage of the method developed by Park et al. in [34] is related to designing a stabilizing controller. For a controller design, a sort of congruent transformation is needed. The e_i matrices do not allow using such a transformation .

A summary of the method in [34]:

- it does not make any bounding;
- it uses delay of the type $0 \leq \tau \leq \rho$, $\dot{\tau} \leq \mu$;
- it uses 4 Lyapunov terms;
- it uses 5 parameter matrices;
- it uses 7 free-weighting matrices; and
- it considers no uncertainties.

Yong et al. in [58] used the new term introduced in [34]. By including the new term Yong et al. used $x(t), x(t-\tau)$ and $x(t-\rho)$ as states in the LMI. Yong et al. showed that their method gives better results than [56].

A summary of the method in [58]:

- it does not make any bounding;
- it uses delay of the type $0 \leq \tau \leq \rho$ $\dot{\tau} \leq \mu$;
- it uses 4 Lyapunov terms;
- it uses 5 parameter matrices;
- it uses 9 free-weighting matrices; and
- it considers polytypic and norm-bounded uncertainties.

Jiang et al. in [50] considered an interval type delay, which is described by:

$$h_1 \leq \tau \leq h_2$$

In the paper, Jiang et al. avoided the transformation methods which were being used earlier. During the derivation, upper-bounding was used to tackle some terms. No constraint was put on the delay rate of change to allow fast dynamics.

A summary of the method in [50]:

- it does not include any model transformation;
- it contains upper-bounding;
- it uses delay of the type $h_1 \leq \tau \leq h_2$;
- it uses 4 Lyapunov terms;

- it uses 4 parameter matrices;
- it uses 3 free-weighting matrices; and
- it considers norm-bounded uncertainties.

Yong et al. in [59] followed a very similar approach to the one developed in [50]. They considered the following delay type:

$$\begin{aligned} h_1 &\leq \tau \leq h_2 \\ \dot{\tau} &\leq \mu \end{aligned}$$

Considering this delay type opens a new way to look at the problem and gives results that are very general. In the paper [59], μ is no longer bounded by 1, and the criticism of fast dynamic is resolved. Yong et al. showed that the method used in [50] ignores useful terms, which leads to greater conservatism. A better result than [50] is obtained in the paper.

A summary of the method in [59]:

- it does not include any model transformation;
- it does not make any bounding;
- it uses delay of the type; $h_1 \leq \tau \leq h_2$, $\dot{\tau} \leq \mu$;
- it uses 5 Lyapunov terms;
- it uses 5 parameter matrices;
- it uses 6 free-weighting matrices; and
- it does not consider any type of uncertainties.

3.3 Comparison and Comments

From the methods discussed in the chapter, six were selected for evaluation. These methods are:

1. Fridman et al. in [3];
2. Min et al. in [32];
3. He et al. in [56];
4. Jing et al. in [51];
5. Lin et al. in [36]; and
6. Yong et al. in [58].

These methods deal with time varying delay, with an upper bound in the delay rate of change. In these methods, the developed LMIs can be extended for state-feedback controller design. The following example was used in most of the papers discussed earlier; therefore, we use it as a reference for comparison:

$$A = \begin{bmatrix} -2.0 & 0.0 \\ 0.0 & -0.9 \end{bmatrix} \quad A_d = \begin{bmatrix} -1.0 & 0.0 \\ -1.0 & -1.0 \end{bmatrix} \tag{3.20}$$

Fixed values were given to μ. The largest value of ρ (ρ_{max}) with which the system 3.20 is proved by each of these methods is found. For every value of μ the corresponding ρ_{max}s were reported. The selected values of μ were 0, 0.5, 0.9 and 3. Table 3.1 shows the obtained results. One maybe surprised to see exactly the same results from different methods. This happened because they are using the same Lyapunov functional. But [58] has exceptional results because the Lyapunov functional there has an additional term. This term gives the method in [58] an advantage over others.

Method	$\mu = 0$	$\mu = 0.5$	$\mu = 0.9$	$\mu = 3$
[3]	4.47	2.0	1.180	X
[32]	4.472	2.008	1.180	X
[56]	4.472	2.008	1.180	.999
[51]	4.472	2.008	1.180	.999
[36]	4.472	2.008	1.180	X
[58]	4.472	2.0430	1.3780	1.3450

TABLE 3.1: Comparison between previous methods for the stability of systems with time delay

From this table it is clear that all the methods except [58] are almost the same. For this reason, the results found in the present thesis are going to be compared with the method in [58] only. In the present thesis, additional terms will be added to calculate the H_∞ gain from the disturbance to the controlled output. Also, the developed methods should be extendible for state-feedback controller design. Since the introduction of polytypic or norm-bounded uncertainties in the LMI is straightforward, only the polytypic type will be considered. The norm-bounded uncertainty can easily be involved in the LMIs.

CHAPTER 4

FIRST APPROACH

This chapter introduces a further development of stability and feedback stabilization of linear time-delay (LTD) systems. Specifically, a Lyapunov-Krasovskii functional (LKF) is constructed with compensatory terms for the enlarged integration time-span, utilizing a smaller number of LMI decision variables. The considered time-delay factor is a differentiable time-varying function satisfying some bounding relations. An $\mathcal{L}_2$-performance analysis is used to derive the solution for nominal and polytypic models. Other existing results are compared so as to demonstrate the potential of our methodology. A robust performance synthesis is then done to design feedback schemes, based on state and dynamic output feedback schemes. All the developed results guarantee that the corresponding nominally-linear system enjoys the delay-dependent robust stability with an $\mathcal{L}_2-$gain smaller than a prescribed constant level. The results are expressed in terms of convex optimization over LMIs and tested on a representative example.

4.1 Problem Statement

Consider the following class of the linear nominal time-delay (LNTD) system:

$$\begin{aligned}\dot{x}(t) &= A_o x(t) + A_{do} x(t-\tau) + B_o u(t) + \Gamma_o w(t) \\ y(t) &= C_o x(t) + C_{do} x(t-\tau) + F_o u(t) + \Psi_o w(t) \\ z(t) &= G_o x(t) + G_{do} x(t-\tau) + D_o u(t) + \Phi_o w(t) \end{aligned} \quad (4.1)$$

where $x(t) \in \Re^n$ is the state vector, $y(t) \in \Re^p$ is the measured output, $u(t) \in \Re^m$ is the control input, $w(t) \in \Re^q$ is the disturbance input which belongs to $\mathcal{L}_2[0,\infty)$ and $z(t) \in \Re^q$ is the controlled output. The initial condition (ϕ) is a differentiable vector-valued function on $[-\tau, 0]$ where $\tau > 0$ is a time-delay factor. The matrices $A_o \in \Re^{nn}$, $B_o \in \Re^{nm}$, $G_o \in \Re^{qn}$, $D_o \in \Re^{qm}$, $A_{do} \in \Re^{nn}$, $\Phi_o \in \Re^{qq}$, $\Gamma_o \in \Re^{nq}$, $C_o \in \Re^{pn}$, $C_{do} \in \Re^{pn}$, $F_o \in \Re^{pm}$, $\Psi_o \in \Re^{pq}$ are real and known constant matrices.

In the sequel, the delay $\tau(t)$ is assumed to be a differentiable time-varying function satisfying $0 < \tau(t) \le \varrho, \quad \dot{\tau}(t) \le \mu$ where the bounds ϱ and μ are known constant scalars. It is remarked that the usual bounding relation $\mu < 1$ [61, 27, 60] is not required in the present work.

Our purpose is to develop robust criteria for delay-dependent asymptotic stability and stabilization of the system 4.1 with a prescribed performance measure.

4.2 Delay-Dependent $\mathcal{L}_2$ Gain Analysis

In this section, we develop a new criterion for LMI-based characterization of delay-dependent asymptotic stability and $\mathcal{L}_2-$gain analysis. The criterion includes some free-weighting matrices aims at expanding the range of applicability of the developed conditions. The following theorem establishes the main result for the LNTD system:

Theorem 4.1 *Given $\varrho > 0$ and $\mu > 0$. The system 4.1 with $u(.) \equiv 0$ is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ if there exist symmetric matrices $0 < \mathcal{P}$, $0 < \mathcal{W}_a$, $0 < \mathcal{W}_c$, $0 < \mathcal{Q}$, $0 < \mathcal{R}$, weighting matrices $N_a, N_c, N_s, M_a, M_c, M_s$ and a scalar $\gamma > 0$ satisfying the following LMI:*

$$\Xi = \begin{bmatrix} \Xi_o & \varrho\mathcal{M} & \varrho\mathcal{N} & \Xi_x \\ \bullet & -\varrho\mathcal{W}_a & 0 & 0 \\ \bullet & \bullet & -\varrho\mathcal{W}_c & 0 \\ \bullet & \bullet & \bullet & -\Xi_y \end{bmatrix} < 0 \tag{4.2}$$

where

$$\begin{aligned}
\Xi_o &= \begin{bmatrix} \Xi_{o1} & \Xi_{o2} & \Xi_{o3} \\ \bullet & \Xi_{o4} & \Xi_{o5} \\ \bullet & \bullet & \Xi_{o6} \end{bmatrix} \\
\Xi_{o1} &= \mathcal{P}A_o + A_o^t\mathcal{P} + \mathcal{Q} + \mathcal{R} + N_a + N_a^t + M_a + M_a^t \\
\Xi_{o2} &= \mathcal{P}A_{do} - 2N_a + N_c^t + M_c^t, \quad \mathcal{W} = \mathcal{W}_a + \mathcal{W}_c \\
\Xi_{o3} &= N_a - M_a + N_s^t + M_s^t \\
\Xi_{o4} &= -(1-\mu)\mathcal{Q} - 2N_c - 2N_c^t, \\
\Xi_{o5} &= N_c - M_c - 2N_s^t \\
\Xi_{o6} &= -\mathcal{R} + N_s + N_s^t - M_s - M_s^t \\
\Xi_y &= \begin{bmatrix} \gamma^2 I & -\Phi_o^t & -\varrho\Gamma_o^t\mathcal{W} \\ \bullet & I & 0 \\ \bullet & \bullet & \varrho\mathcal{W} \end{bmatrix}, \ \mathcal{N} = \begin{bmatrix} N_a \\ N_c \\ N_s \end{bmatrix},
\end{aligned} \tag{4.3}$$

$$\mathcal{M} = \begin{bmatrix} M_a \\ M_c \\ M_s \end{bmatrix}, \ \Xi_x = \begin{bmatrix} \mathcal{P}\Gamma_o & G_o^t & \varrho A_o^t \mathcal{W} \\ 0 & G_{do}^t & \varrho A_{do}^t \mathcal{W} \\ 0 & 0 & 0 \end{bmatrix} \tag{4.4}$$

Proof : In terms of $\xi(t) = \begin{bmatrix} x^t(t) & x^t(t-\tau(t)) & x^t(t-\varrho) \end{bmatrix}^t$ and using the classical Leibniz rule $x(t-\theta) = x(t) - \int_{t-\theta}^{t} \dot{x}(s)ds$ for any matrices $N_a,\ N_c,\ N_s,\ M_a,\ M_c,\ M_s$ of appropriate dimensions, the following equations hold:

$$2\ \xi^t(t)2\mathcal{N}\left[-\int_{t-\tau(t)}^{t} \dot{x}(s)ds + x(t) - x(t-\tau) \right] = 0 \tag{4.5}$$

$$2\ \xi^t(t)(\mathcal{M}-\mathcal{N})\left[-\int_{t-\varrho}^{t} \dot{x}(s)ds + x(t) - x(t-\varrho) \right] = 0 \tag{4.6}$$

Expansion of 4.5-4.6 gives:

$$\begin{aligned} & x^t(t)[N_a + N_a^t + M_a + M_a^t]x(t) + 2x^t(t)[-2N_a + M_c^t + N_c^t]x(t-\tau(t)) \\ & +2x^t(t)[N_a - M_a + N_s^t + M_s^t]x(t-\varrho) + 2x^t(t-\tau(t))[-2N_c - 2N_c^t]x(t-\tau(t)) \\ & +2x^t(t-\tau(t))[N_c - 2N_s^t - M_c]x(t-\varrho) + 2x^t(t-\varrho)[N_s + N_s^t - M_s - M_s^t]x(t-\varrho) \\ & -2\ \xi^t(t)2\mathcal{N}\int_{t-\tau(t)}^{t} \dot{x}(s)ds - 2\ \xi^t(t)(\mathcal{M}-\mathcal{N})\int_{t-\varrho}^{t} \dot{x}(s)ds = 0 \end{aligned} \tag{4.7}$$

Consider now the augmented Lyapunov-Krasovskii functional (ALKF):

$$\begin{aligned} V(t) &= V_o(t) + V_a(t) + V_c(t) + V_m(t) \\ V_o(t) &= x^t(t)\mathcal{P}x(t),\ V_a(t) = \int_{-\varrho}^{0}\int_{t+s}^{t} \dot{x}^t(\alpha)(\mathcal{W}_a + \mathcal{W}_c)\dot{x}(\alpha)d\alpha\ ds, \\ V_c(t) &= \int_{t-\varrho}^{t} x^t(s)\mathcal{R}x(s)ds,\ V_m(t) = \int_{t-\tau(t)}^{t} x^t(s)\mathcal{Q}x(s)\ ds \end{aligned} \tag{4.8}$$

where $0 < \mathcal{P} = \mathcal{P}^t,\ 0 < \mathcal{W}_a = \mathcal{W}_a^t,\ 0 < \mathcal{W}_c = \mathcal{W}_c^t,\ 0 < \mathcal{Q} = \mathcal{Q}^t,\ 0 < \mathcal{R} = \mathcal{R}^t$ are matrices of appropriate dimensions. The first term in 4.8 is standard to nominal system without delay. The second and fourth terms correspond to the delay-dependent conditions. The third term is introduced to compensate for the enlarged time interval from $t - \varrho \to t$ to $t - \tau \to t$. A straightforward computation gives the time-derivative of $V(x)$ along the solutions of 4.1 with $w(t) \equiv 0$ as:

$$\begin{aligned}
\dot{V}_o(t) &= 2x^t(t)\mathcal{P}\dot{x}(t) \\
&= 2x^t(t)\mathcal{P}[A_o x(t) + A_{do}x(t-\tau)] \\
\dot{V}_a(t) &= \varrho\, \dot{x}^t(t)(\mathcal{W}_a + \mathcal{W}_c)\dot{x}(t) - \int_{t-\varrho}^{t} \dot{x}^t(s)(\mathcal{W}_a + \mathcal{W}_c)\dot{x}(s)ds \\
\dot{V}_c(t) &= x^t(t)\mathcal{R}x(t) - x^t(t-\varrho)\mathcal{R}x(t-\varrho) \\
\dot{V}_m(t) &= x^t(t)\mathcal{Q}x(t) - (1-\dot{\tau})\, x^t(t-\tau(t))\mathcal{Q}x(t-\tau(t)) \\
&\leq x^t(t)\mathcal{Q}x(t) - (1-\mu)\, x^t(t-\tau(t))\mathcal{Q}x(t-\tau(t))
\end{aligned} \tag{4.9}$$

From 4.8-4.9 and using 4.7, we have:

$$\begin{aligned}
&\dot{V}(t)|_{4.1} \leq x^t(t)[\mathcal{P}A_o + A_o^t\mathcal{P} + \mathcal{Q} + \mathcal{R} + N_a + N_a^t + M_a + M_a^t]x(t) \\
&+2x^t(t)[\mathcal{P}A_{do} - 2N_a + M_c^t + N_c^t]x(t-\tau) + 2x^t(t)[N_a - M_a + N_s^t + M_s^t]x(t-\varrho) \\
&+2x^t(t-\tau)[N_c - 2N_s^t - M_c]x(t-\varrho) - x^t(t-\tau)[(1-\mu)\mathcal{Q} + 2N_c + 2N_c^t]x(t-\tau(t)) \\
&+x^t(t-\varrho)[-\mathcal{R} + N_s + N_s^t - M_s - M_s^t]x(t-\varrho) + \varrho\, \dot{x}^t(t)(\mathcal{W}_a + \mathcal{W}_c)\dot{x}(t) \\
&- \int_{t-\varrho}^{t} \dot{x}^t(s)(\mathcal{W}_a + \mathcal{W}_c)\dot{x}(s)ds - 2\, \xi^t(t)2\mathcal{N} \int_{t-\tau(t)}^{t} \dot{x}(s)ds \\
&-2\, \xi^t(t)(-\mathcal{N}) \int_{t-\varrho}^{t} \dot{x}(s)ds - 2\, \xi^t(t)\mathcal{M} \int_{t-\varrho}^{t} \dot{x}(s)ds
\end{aligned} \tag{4.10}$$

To manipulate the terms in 4.10, first consider:

$$\begin{aligned} -2 \quad & \xi^t(t)2\mathcal{N}\int_{t-\tau(t)}^{t} \dot{x}(s)ds + 2\,\xi^t(t)\mathcal{N}\int_{t-\varrho}^{t} \dot{x}(s)ds \\ = \quad & -2\,\xi^t(t)\mathcal{N}\int_{t-\tau(t)}^{t} \dot{x}(s)ds + 2\,\xi^t(t)\mathcal{N}\int_{t-\varrho}^{t-\tau(t)} \dot{x}(s)ds \end{aligned} \tag{4.11}$$

and then consider:

$$\begin{aligned} & \int_{t-\varrho}^{t} \dot{x}^t(s)(\mathcal{W}_a+\mathcal{W}_c)\dot{x}(s)ds \\ & = \int_{t-\varrho}^{t} \dot{x}^t(s)\mathcal{W}_a\dot{x}(s)ds + \int_{t-\varrho}^{t} \dot{x}^t(s)\mathcal{W}_c\dot{x}(s)ds \\ & = \int_{t-\varrho}^{t} \dot{x}^t(s)\mathcal{W}_a\dot{x}(s)ds + \int_{t-\tau(t)}^{t} \dot{x}^t(s)\mathcal{W}_c\dot{x}(s)ds \\ & + \int_{t-\varrho}^{t-\tau(t)} \dot{x}^t(s)\mathcal{W}_c\dot{x}(s)ds \end{aligned} \tag{4.12}$$

Then $\dot{V}(t)$ becomes:

$$\begin{aligned} \dot{V}(t)|_{4.1} \quad \leq \quad & \xi^t(t)\Xi_o\xi(t) - \int_{t-\varrho}^{t} \dot{x}^t(s)\mathcal{W}\dot{x}(s)ds \\ + \quad & \xi^t(t)\begin{bmatrix} \varrho A_o^t \\ \varrho A_{do}^t \\ 0 \end{bmatrix}\mathcal{W}\begin{bmatrix} \varrho A_o^t \\ \varrho A_{do}^t \\ 0 \end{bmatrix}^t \xi(t) - 2\xi^t(t)\mathcal{N}\int_{t-\tau(t)}^{t} \dot{x}(s)ds \\ - \quad & 2\xi^t(t)(-\mathcal{N})\int_{t-\varrho}^{t-\tau(t)} \dot{x}(s)ds - 2\xi^t(t)\mathcal{M}\int_{t-\varrho}^{t} \dot{x}(s)ds \end{aligned} \tag{4.13}$$

Consider adding and subtracting the terms:

$$\xi^t(t)\Big\{\varrho\mathcal{M}\mathcal{W}_a^{-1}\mathcal{M}^t + \varrho\mathcal{N}\mathcal{W}_c^{-1}\mathcal{N}^t\Big\}\xi \tag{4.14}$$

Now consider the following terms:

$$
\begin{aligned}
&\varrho\xi^t(t)\,\mathcal{M}\,\mathcal{W}_a^{-1}\,\mathcal{M}^t\,\xi(t)+\varrho\xi^t(t)\,\mathcal{N}\,\mathcal{W}_c^{-1}\,\mathcal{N}^t\,\xi(t)\\
&-\varrho\xi^t(t)\,\mathcal{M}\,\mathcal{W}_a^{-1}\,\mathcal{M}^t\,\xi(t)-\tau(t)\xi^t(t)\,\mathcal{N}\,\mathcal{W}_c^{-1}\,\mathcal{N}^t\,\xi(t)\\
&-(\varrho-\tau(t))\xi^t(t)\,\mathcal{N}\,\mathcal{W}_c^{-1}\,\mathcal{N}^t\,\xi(t)\\
&-2\,\xi^t(t)\mathcal{N}\int_{t-\tau(t)}^{t}\dot{x}(s)ds+2\,\xi^t(t)\mathcal{N}\int_{t-\varrho}^{t-\tau(t)}\dot{x}(s)ds\\
&-2\,\xi^t(t)\mathcal{M}\int_{t-\varrho}^{t}\dot{x}(s)ds+\int_{t-\varrho}^{t-\tau(t)}\dot{x}^t(s)\mathcal{W}_c\dot{x}(s)ds\\
&+\int_{t-\tau(t)}^{t}\dot{x}^t(s)\mathcal{W}_c\dot{x}(s)ds+\int_{t-\varrho}^{t}\dot{x}^t(s)\mathcal{W}_a\dot{x}(s)ds
\end{aligned}
\tag{4.15}
$$

The terms in 4.15 after some manipulation become:

$$
\begin{aligned}
&=\xi^t(t)\Big\{\varrho\mathcal{M}\mathcal{W}_a^{-1}\mathcal{M}^t+\varrho\mathcal{N}\mathcal{W}_c^{-1}\mathcal{N}^t\Big\}\xi-\int_{t-\tau(t)}^{t}[\xi^t\mathcal{N}+\dot{x}^t\mathcal{W}_c]\mathcal{W}_c^{-1}[\xi^t\mathcal{N}+\dot{x}^t\mathcal{W}_c]^t ds\\
&-\int_{t-\varrho}^{t-\tau(t)}[-\xi^t\mathcal{N}+\dot{x}^t\mathcal{W}_c]\mathcal{W}_c^{-1}[-\xi^t\mathcal{N}+\dot{x}^t\mathcal{W}_c]^t ds\\
&-\int_{t-\varrho}^{t}[\xi^t\mathcal{M}+\dot{x}^t\mathcal{W}_a]\mathcal{W}_a^{-1}[\xi^t\mathcal{M}+\dot{x}^t\mathcal{W}_a]^t ds
\end{aligned}
\tag{4.16}
$$

Further manipulations of 4.13 result in:

$$
\begin{aligned}
\dot{V}(t)|_{4.1}\ &\le\ \xi^t(t)[\Xi_o+\varrho\mathcal{M}\mathcal{W}_a^{-1}\mathcal{M}^t+\tau(t)\mathcal{N}\mathcal{W}_c^{-1}\mathcal{N}^t\\
&+\ (\varrho-\tau(t))\mathcal{N}\mathcal{W}_c^{-1}\mathcal{N}^t]\xi(t)+\varrho\,\dot{x}^t(t)(\mathcal{W}_a+\mathcal{W}_c)\dot{x}(t)\\
&-\ \int_{t-\tau(t)}^{t}[\xi^t\mathcal{N}+\dot{x}^t\mathcal{W}_c]\mathcal{W}_c^{-1}[\xi^t\mathcal{N}+\dot{x}^t\mathcal{W}_c]^t ds\\
&-\ \int_{t-\varrho}^{t-\tau(t)}[-\xi^t\mathcal{N}+\dot{x}^t\mathcal{W}_c]\mathcal{W}_c^{-1}[-\xi^t\mathcal{N}+\dot{x}^t\mathcal{W}_c]^t ds\\
&-\ \int_{t-\varrho}^{t}[\xi^t\mathcal{M}+\dot{x}^t\mathcal{W}_a]\mathcal{W}_a^{-1}[\xi^t\mathcal{M}+\dot{x}^t\mathcal{W}_a]^t ds\\
\le\xi^t(t)[\Xi_o\ &+\ \varrho\mathcal{M}\mathcal{W}_a^{-1}\mathcal{M}^t+\varrho\mathcal{N}\mathcal{W}_c^{-1}\mathcal{N}^t]\xi(t)+\varrho\,\dot{x}^t(t)(\mathcal{W}_a+\mathcal{W}_c)\dot{x}(t)
\end{aligned}
\tag{4.17}
$$

In view of 4.2 with $G_o \equiv 0,\ G_d \equiv 0,\ \Gamma_o \equiv 0,$ and Schur's complements, it follows from 4.17 that $\dot{V}(t)|_{4.1} < 0$ which establishes the internal asymptotic stability.

Consider the performance measure $J = \int_0^\infty \Bigg(z^t(s)z(s) - \gamma^2 w^t(s)w(s) \Bigg) ds.$ For any $w(t) \in \mathcal{L}_2(0,\infty) \neq 0$ and zero initial condition $x(0) = 0$, we have:

$$
\begin{aligned}
J &= \int_0^\infty \left(z^t(s)z(s) - \gamma^2 w^t(s)w(s) \ + \dot{V}(x)|_{4.1} \right) ds - \dot{V}(x)|_{4.1} \\
&\leq \int_0^\infty \left(z^t(s)z(s) - \gamma^2 w^t(s)w(s) \ + \dot{V}(x)|_{4.1} \right) ds
\end{aligned}
$$

Proceeding as before, we get:

$$
\begin{aligned}
& z^t(s)z(s) - \gamma^2 w^t(s)w(s) \ + \dot{V}(s)|_{4.1} = \bar{\chi}^t(s)\ \bar{\Xi}\ \bar{\chi}(s), \\
& \bar{\chi}(s) = \begin{bmatrix} x^t(s) & x^t(s-\tau(t)) & x^t(t-\varrho) & w(s) \end{bmatrix}^t
\end{aligned} \tag{4.18}
$$

where $\bar{\Xi}$ corresponds to Ξ_ϱ in 4.2 by Schur's complements. It is readily seen from 4.2 that:

$$
z^t(s)z(s) - \gamma^2 w^t(s)w(s) \ + \dot{V}(s)|_{4.1} < 0
$$

for arbitrary $s \in [t, \infty)$, which implies for any $w(t) \in \mathcal{L}_2(0,\infty) \neq 0$ that $J < 0$ leading to $||z(t)||_2 \ < \ \gamma\ ||w(t)||_2$ and the proof is completed.

4.2.1 Linear Uncertain Systems

Suppose now that the system 4.1 has the state-space model:

$$
\begin{aligned}
\dot{x}(t) &= A_\Delta x(t) + A_{d\Delta} x(t-\tau) + B_\Delta u(t) + \Gamma_\Delta w(t) \\
y(t) &= C_\Delta x(t) + C_{d\Delta} x(t-\tau) + F_\Delta u(t) + \Psi_\Delta w(t) \\
z(t) &= G_\Delta x(t) + G_{d\Delta} x(t-\tau) + D_\Delta u(t) + \Phi_\Delta w(t)
\end{aligned} \tag{4.19}
$$

whose matrices contain uncertainties which belong to a real convex bounded polytypic model of the type:

$$
\begin{aligned}
&\begin{bmatrix} A_\Delta & A_{d\Delta} & B_\Delta & \Gamma_\Delta \\ C_\Delta & C_{d\Delta} & F_\Delta & \Psi_\Delta \\ G_\Delta & G_{d\Delta} & D_\Delta & \Phi_\Delta \end{bmatrix} \in \Pi_\lambda \\
&\triangleq \left\{ \begin{bmatrix} A_\lambda & A_{d\lambda} & B_\lambda & \Gamma_\lambda \\ C_\lambda & C_{d\lambda} & F_\lambda & \Psi_\lambda \\ G_\lambda & G_{d\lambda} & D_\lambda & \Phi_\lambda \end{bmatrix} = \sum_{j=1}^{N} \lambda_j \begin{bmatrix} A_j & A_{dj} & B_j & \Gamma_j \\ C_j & C_{dj} & F_j & \Psi_j \\ G_j & G_{dj} & D_j & \Phi_j \end{bmatrix}, \lambda_j \in \Lambda \right\} \qquad (4.20)
\end{aligned}
$$

where Λ is the unit simplex,

$$
\Lambda \triangleq \left\{ (\lambda_1, \cdots, \lambda_N) : \sum_{j=1}^{N} \lambda_j = 1 \ , \ \lambda_j \geq 0 \right\} \qquad (4.21)
$$

Define the vertex set $\mathcal{N} = \{1, ..., N\}$. We use $\{A_o, ..., \Phi_o\}$ to imply generic system matrices and $\{A_{oj}, ..., \Phi_{oj}, \ j \in \mathcal{N}\}$ to represent the respective values at the vertices. It is a straightforward task to show that the following result holds:

Theorem 4.2 *The system 4.1 with $u(.) \equiv 0$ and polytypic representation 4.20-4.21 is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ if there exist symmetric matrices $\mathcal{P}, \ \mathcal{Q}, \ \mathcal{W}$, weighting matrices $\Theta, \ \Upsilon$ and a scalar γ satisfying:*

$$
\Xi_j = \begin{bmatrix} \Xi_{oj} & \varrho\mathcal{M} & \varrho\mathcal{N} & \Xi_{xj} \\ \bullet & -\varrho\mathcal{W}_a & 0 & 0 \\ \bullet & \bullet & -\varrho\mathcal{W}_c & 0 \\ \bullet & \bullet & \bullet & -\Xi_{yj} \end{bmatrix} < 0 \qquad (4.22)
$$

where $j = 1, ..., N$ *and:*

$$
\begin{aligned}
\Xi_{oj} &= \begin{bmatrix} \Xi_{1j} & \Xi_{2j} & \Xi_{o3} \\ \bullet & \Xi_{o4} & \Xi_{o5} \\ \bullet & \bullet & \Xi_{o6} \end{bmatrix} \\
\Xi_{1j} &= \mathcal{P}A_{oj} + A_{oj}^t\mathcal{P} + \mathcal{Q} + \mathcal{R} + N_a + N_a^t + M_a + M_a^t \\
\Xi_{2j} &= \mathcal{P}A_{dj} - 2N_a + N_c^t + M_c^t \\
\Xi_{yj} &= \begin{bmatrix} \gamma^2 I & -\Phi_{oj}^t & -\varrho\Gamma_{oj}^t\mathcal{W} \\ \bullet & I & 0 \\ \bullet & \bullet & \varrho\mathcal{W} \end{bmatrix}, \\
\Xi_{xj} &= \begin{bmatrix} \mathcal{P}\Gamma_{oj} & G_{oj}^t & \varrho A_{oj}^t\mathcal{W} \\ 0 & G_{dj}^t & \varrho A_{dj}^t\mathcal{W} \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{4.23}
$$

and $\Xi_{o3},\ \Xi_{o4},\ \Xi_{o5},\ \Xi_{o6}$ *are given in 4.4.*

Remark 4.2.1 *It is important to recognize that our method provides a substantial improvement over the recently developed method of [58]. Hence it is expected to yield the least conservative delay-dependent stability results in terms of two aspects. One aspect would be due to reduced computational load as evidenced by a simple comparison with the fewer manipulated variables. Another aspect arises by noting that LMIs 4.2 and 4.4 theoretically cover the results of [51, 61, 52] as special cases. Furthermore, in the absence of delay* $A_d \equiv 0,\ \mathcal{Q} \equiv 0,\ \mathcal{W} \equiv 0$*, it is easy to infer that LMIs 4.2 and 4.4 will eventually reduce to a parameterized delay-free criteria.*

4.2.2 Example 4.1

An open-loop stable time-delay system for a chemical reactor is considered here [48]. In the reactor, raw materials A and B take part in three chemical reactions that produce a product P along with some other byproducts (like C). By linearization and time scaling, the state variables are the deviations from the nominal values: in the weight compositions of reactant A, of reactant B, of intermediate product C and of product P. The control variables are relative deviations in the feed rates. Using typical values [48], the system matrices are:

$$
\begin{aligned}
A_o &= \begin{bmatrix} -4.93 & -1.01 & 0 & 0 \\ -3.2 & -5.3 & -12.8 & 0 \\ 6.4 & 0.347 & -32.5 & -1.04 \\ 0 & 0.833 & 11.0 & -3.96 \end{bmatrix}, \; \Gamma_o = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}, \\
A_{do} &= \begin{bmatrix} 1.92 & 0 & 0 & 0 \\ 0 & 1.92 & 0 & 0 \\ 0 & 0 & 1.87 & 0 \\ 0 & 0 & 0 & 0.724 \end{bmatrix}, \Phi = 0.1 \\
G_o &= [0.1 \;\; 0.2 \;\; 0.4 \;\; 0.3], \; G_{do} = [0.01 \;\; 0 \;\; 0.01 \;\; 0]
\end{aligned}
$$

In terms of the number of free variables N_v, the average time to complete one LMI iteration T_a, the total elapsed time T_e to reach a desirable ϱ and the maximum ϱ, a sequence of numerical experiments is performed on a standard computing facility (*Pentium 4 CPU- 3 G Hz processor with 512 Mb RAM using Matlab 7*). Table 4.1 contains a summary of the computational results of our methods as compared to the one in [58]. From the table, it is clear that our method is less conservative, has fewer variables, and requires less computation time. The resulting open loop state responses

Method	N_v	T_a	T_e	ϱ
[58]	204	1.7765 s	116.843 s	0.652
Theorem 4.1	146	0.9234 s	44.969 s	0.874

TABLE 4.1: First approach computational summary with $\mu = 2$: Example 4.1

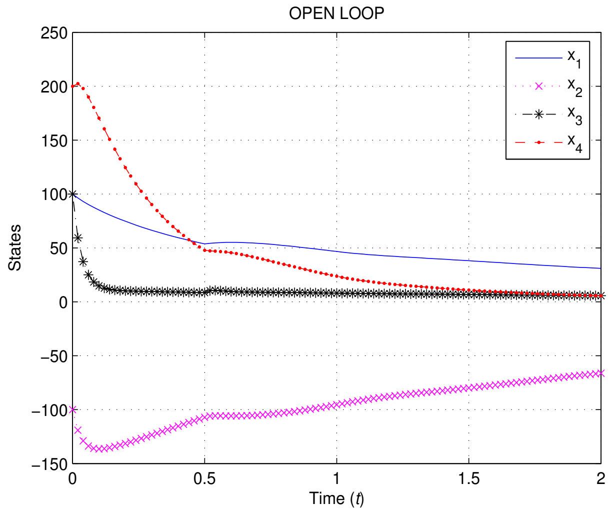

Figure 4.1: Open-loop state-trajectories: Example 4.1

are plotted in Figure 4.1.

4.3 State-Feedback Stabilization

To study state-feedback stabilization, consider apply the state-feedback control $u(t) = K_s x(t)$ to the nominal system 4.1 and define $A_s = A_o + B_o K_s$ and $G_s = G_o + D_o K_s$.

It follows from **Theorem 4.1** that the resulting closed-loop system is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ if there exist symmetric matrices $0 < \mathcal{P}$, $0 < \mathcal{W}_a$, $0 < \mathcal{W}_c$, $0 < \mathcal{Q}$, $0 < \mathcal{R}$, weighting matrices

$N_a,\ N_c,\ N_s,\ M_a,\ M_c,\ M_s$ and a scalar $\gamma > 0$ satisfying the following LMI:

$$\Xi_s = \begin{bmatrix} \Xi_{os} & \varrho\mathcal{M} & \varrho\mathcal{N} & \Xi_{xs} \\ \bullet & -\varrho\mathcal{W}_a & 0 & 0 \\ \bullet & \bullet & -\varrho\mathcal{W}_c & 0 \\ \bullet & \bullet & \bullet & -\Xi_y \end{bmatrix} < 0 \tag{4.24}$$

$$\begin{aligned} \Xi_{os} &= \begin{bmatrix} \Xi_{1s} & \Xi_{02} & \Xi_{o3} \\ \bullet & \Xi_{o4} & \Xi_{o5} \\ \bullet & \bullet & \Xi_{o6} \end{bmatrix} \\ \Xi_{1s} &= \mathcal{P}A_s + A_s^t\mathcal{P} + \mathcal{Q} + \mathcal{R} + N_a + N_a^t + M_a + M_a^t \\ \Xi_{xs} &= \begin{bmatrix} \mathcal{P}\Gamma_o & G_s^t & \varrho A_s^t\mathcal{W} \\ 0 & G_{do}^t & \varrho A_{do}^t\mathcal{W} \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{4.25}$$

The main design result is established by the following theorem:

Theorem 4.3 *Given scalars ϱ and μ, the system 4.1 with $u(t) = K_s x(t)$ is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ if there exist symmetric matrices $\mathcal{X} > 0$, $\{\Theta\}_1^2 > 0$ a matrix $\mathcal{Y}$, weighting matrices $\{\Theta\}_3^8$ and scalars $\gamma > 0$, $\beta_a > 0$ and $\beta_c > 0$ satisfying the following LMI:*

$$\begin{bmatrix} \Pi_o & \varrho\widehat{\Theta} & \varrho\widetilde{\Theta} & \Pi_v \\ \bullet & -\varrho\mathcal{Z} & 0 & 0 \\ \bullet & \bullet & -\varrho\mathcal{G} & 0 \\ \bullet & \bullet & \bullet & -\Pi_w \end{bmatrix} < 0 \tag{4.26}$$

$$
\begin{aligned}
\Pi_o &= \begin{bmatrix} \Pi_{o1} & \Pi_{o2} & \Pi_{o3} \\ \bullet & \Pi_{o4} & \Pi_{o5} \\ \bullet & \bullet & \Pi_{o6} \end{bmatrix} \\
\Pi_{o1} &= A_o\mathcal{X} + \mathcal{X}A_o^t + B_o\mathcal{Y} + \mathcal{Y}^tB_o^t + \Theta_1 + \Theta_2 + \Theta_3 + \Theta_3^t + \Theta_6 + \Theta_6^t \\
\Pi_{o2} &= A_{do}\mathcal{X} - 2\Theta_3 + \Theta_4^t + \Theta_7^t,\ \ \mathcal{Z} = \beta_a\mathcal{X},\ \mathcal{G} = \beta_c\mathcal{X}, \\
\Pi_a &= (\mathcal{Z} + \mathcal{G})A_o^t + (\beta_a + \beta_c)\mathcal{Y}^tB_o^t \\
\Pi_{o3} &= \Theta_3 - \Theta_6 + \Theta_5^t + \Theta_8^t,\ \Pi_c = \mathcal{X}G_o^t + \mathcal{Y}^tD_o^t \\
\Pi_{o4} &= -(1-\mu)\Theta_1 - 2\Theta_4 - 2\Theta_4^t,\ \Pi_{o5} = \Theta_4 - \Theta_7 - 2\Theta_5^t \\
\Pi_{o6} &= -\Theta_2 + \Theta_5 + \Theta_5^t - \Theta_8 - \Theta_8^t, \\
\Pi_v &= \begin{bmatrix} \Gamma_o & \Pi_c & \varrho\Pi_a \\ 0 & \mathcal{X}G_{do}^t & \varrho(\mathcal{Z}+\mathcal{G})A_{do}^t \\ 0 & 0 & 0 \end{bmatrix},\ \widetilde{\Theta} = \begin{bmatrix} \Theta_3 \\ \Theta_4 \\ \Theta_5 \end{bmatrix}, \\
\widehat{\Theta} &= \begin{bmatrix} \Theta_6 \\ \Theta_7 \\ \Theta_8 \end{bmatrix},\ \Pi_w = \begin{bmatrix} \gamma^2 I & -\Phi_o^t & -\varrho\Gamma_o^t \\ \bullet & I & 0 \\ \bullet & \bullet & \varrho\mathcal{Z} + \varrho\mathcal{G} \end{bmatrix}
\end{aligned}
\tag{4.27}
$$

Moreover, the gain matrix is given by $K_s = \mathcal{Y}\mathcal{X}^{-1}$.

Proof: Define $\mathcal{X} = \mathcal{P}^{-1}$ and apply the congruent transformation:

$$
\mathcal{T} = diag[\,\mathcal{X},\ \mathcal{X},\ \mathcal{X},\ \mathcal{X},\ \mathcal{X},\ T_2],\ \ T_2 = diag\begin{bmatrix} I & I & I \end{bmatrix}
$$

After making this multiplication the following nonlinear terms appear:

$$
\mathcal{X}\mathcal{W}_a\mathcal{X},\ \mathcal{X}\mathcal{W}_c\mathcal{X},\ \varrho\mathcal{X}A\mathcal{W}_a\ etc
$$

One way to avoid these nonlinearities is to assume $\mathcal{W}_a = \mathcal{W}_c = \mathcal{X}^{-1}$. This is very conservative and so a better choice is to assume:

$$\mathcal{W}_a = \beta_a \mathcal{X}^{-1} = \beta_a P \;\; \mathcal{W}_c = \beta_c \mathcal{X}^{-1} = \beta_c P$$

where β_a and β_c are scalars that can take any positive values. If the system is stable, one can guess the suitable values for β_a and β_c after applying the stability theorem and observing the values of P, $\mathcal{W}_a$ and $\mathcal{W}_c$. If the system is not stable, suitable values for them can be found by trying different values of β_a and β_c. One can make initial guesses by checking different values of β_a and β_c and seeing how fast the LMI passes the feasibility test. The faster the result, the farther these values are from the valid sets of the LMI in 4.24. The matrices in the theorem can be found by the linearizations:

$$\begin{aligned}
\mathcal{Z} &= \beta_a \mathcal{X},\ \mathcal{G} = \beta_c \mathcal{X},\ \Theta_1 = \mathcal{X}\mathcal{Q}\mathcal{X},\ \Theta_2 = \mathcal{X}\mathcal{R}\mathcal{X}, \\
\Theta_3 &= \mathcal{X} N_a \mathcal{X}\ \Theta_4 = \mathcal{X} N_c \mathcal{X},\ \Theta_5 = \mathcal{X} N_s \mathcal{X}, \\
\Theta_6 &= \mathcal{X} M_a \mathcal{X},\ ,\Theta_7 = \mathcal{X} M_c \mathcal{X},\ \Theta_8 = \mathcal{X} M_s \mathcal{X},\ \mathcal{Y} = K_s \mathcal{X},
\end{aligned}$$

and the matrix definitions 4.27, and we obtain LMI 4.26 by Schur's complements.

4.3.1 Example 4.2

We consider the chemical reactor of Example 4.1 with:

$$B_o^t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D_o = [0.4 \;\; 0.2]$$

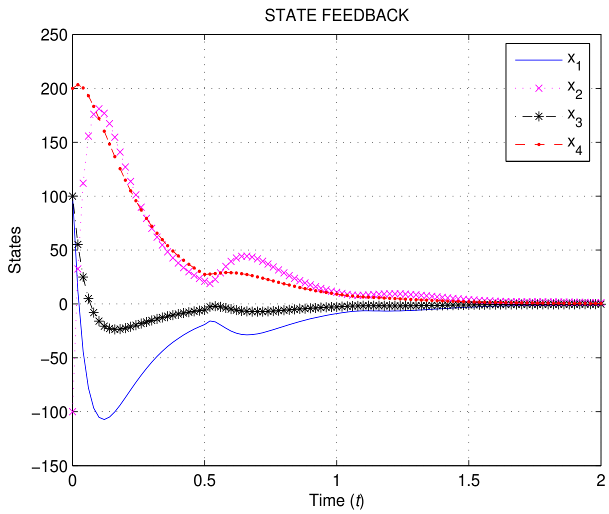

Figure 4.2: State-trajectories by state-feedback: Example 4.2

Application of Theorem 4.3, using the solver LMI-Toolbox, yields the solution:

$$
\begin{aligned}
\mu &= 2,\ \varrho = 0.775,\ \gamma = 0.0160,\ \beta_a = 0.819,\ \beta_c = 0.081 \\
K_s &= \begin{bmatrix} 7.5351 & 1.7396 & -49.5242 & -3.9617 \\ -30.5670 & -10.1238 & 110.4315 & 2.4973 \end{bmatrix}
\end{aligned}
$$

The corresponding closed loop responses are plotted in Figure 4.2. The benefit of applying feedback control is obvious by the well-damped behavior of the closed-loop trajectories.

4.3.2 Dynamic Output-feedback

In the sequel, we consider stabilizing the system 4.1 by means of the dynamic output-feedback controller. The controller is on the following observer:

$$
\begin{aligned}
\dot{x_c}(t) &= A_o x_c(t) + B_o\, u(t) + K_o[y(t) - C_o x_c(t)], \\
u(t) &= K_c x_c \qquad (4.28)
\end{aligned}
$$

Appending the system 4.1 to controller 4.28, we get the closed-loop time-delay (CLTD) system:

$$
\begin{aligned}
\dot{\widetilde{x}}(t) &= \widetilde{A}_o x(t) + \widetilde{A}_d x(t-\tau) + \widetilde{\Gamma}_o w(t), \\
\widetilde{z}(t) &= \widetilde{G}_o x(t) + \widetilde{G}_d x(t-\tau) + \Phi_o w(t)
\end{aligned} \tag{4.29}
$$

where K_o and K_c are the unknown gain matrices to be determined and:

$$
\begin{aligned}
\widetilde{x} &= \begin{bmatrix} x^t(t) & x_c^t(t) \end{bmatrix}^t \quad \widetilde{G}_d = \begin{bmatrix} G_{do} & 0 \end{bmatrix} \\
\widetilde{A}_o &= \begin{bmatrix} A_o & B_o K_c \\ K_o C_o & A_o + B_o K_c - K_o(C_o - F_o K_c) \end{bmatrix}, \ \widetilde{\Gamma}_o = \begin{bmatrix} \Gamma_o \\ K_o \Psi_o \end{bmatrix} \\
\widetilde{A}_d &= \begin{bmatrix} A_{do} & 0 \\ K_o C_{do} & 0 \end{bmatrix}, \ \widetilde{G}_o = \begin{bmatrix} G_o & D_o K_c \end{bmatrix},
\end{aligned} \tag{4.30}
$$

It follows from Theorem 4.1 that, for the given scalars $\mu > 0$, $\varrho > 0$, the system 4.29 is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ if there exist symmetric matrices $\widetilde{P} > 0$, $\widetilde{Q} > 0$, $\widetilde{R} > 0, \widetilde{W}_c > 0$ and $\widetilde{W}_a > 0$ and weighting matrices $\widetilde{N}$, $\widetilde{M}$ and a scalar $\gamma > 0$ satisfying the following LMI:

$$
\Lambda_\varrho = \begin{bmatrix} \Lambda_o & \varrho\widetilde{\mathcal{M}} & \varrho\widetilde{\mathcal{N}} & \Lambda_a \\ \bullet & -\varrho\widetilde{\mathcal{W}}_a & 0 & 0 \\ \bullet & \bullet & -\varrho\widetilde{\mathcal{W}}_c & 0 \\ \bullet & \bullet & \bullet & -\Lambda_s \end{bmatrix} < 0 \tag{4.31}
$$

$$
\Lambda_o = \begin{bmatrix} \Lambda_{o1} & \Lambda_{o2} & \Lambda_{o3} \\ \bullet & \Lambda_{o4} & \Lambda_{o5} \\ \bullet & \bullet & \Lambda_{o6} \end{bmatrix}
$$

$$
\begin{aligned}
\Lambda_{o1} &= \widetilde{\mathcal{P}}\widetilde{A}_o + \widetilde{A}_o^t\widetilde{\mathcal{P}} + \widetilde{\mathcal{Q}} + \widetilde{\mathcal{R}} + \widetilde{N}_a + \widetilde{N}_a^t + \widetilde{M}_a + \widetilde{M}_a^t \\
\Lambda_{o2} &= \widetilde{\mathcal{P}}\widetilde{A}_d - 2\widetilde{N}_a + \widetilde{N}_c^t + \widetilde{M}_c^t,\ \widetilde{\mathcal{W}} = \widetilde{\mathcal{W}}_a + \widetilde{\mathcal{W}}_c \\
\Lambda_{o3} &= \widetilde{N}_a - \widetilde{M}_a + \widetilde{N}_s^t + \widetilde{M}_s^t \\
\Lambda_{o4} &= -(1-\mu)\widetilde{\mathcal{Q}} - 2\widetilde{N}_c - 2\widetilde{N}_c^t,\ \Lambda_{o5} = \widetilde{N}_c - \widetilde{M}_c - 2\widetilde{N}_s^t \\
\Lambda_{o6} &= -\widetilde{\mathcal{R}} + \widetilde{N}_s + \widetilde{N}_s^t - \widetilde{M}_s - \widetilde{M}_s^t \\
\Lambda_s &= \begin{bmatrix} \gamma^2 I & -\Phi_o^t & -\varrho\widetilde{\Gamma}_o\widetilde{\mathcal{W}} \\ \bullet & I & 0 \\ \bullet & \bullet & \varrho\widetilde{\mathcal{W}} \end{bmatrix},\ \widetilde{\mathcal{N}} = \begin{bmatrix} \widetilde{N}_a \\ \widetilde{N}_c \\ \widetilde{N}_s \end{bmatrix}, \\
\widetilde{\mathcal{M}} &= \begin{bmatrix} \widetilde{M}_a \\ \widetilde{M}_c \\ \widetilde{M}_s \end{bmatrix},\ \Lambda_a = \begin{bmatrix} \widetilde{\mathcal{P}}\widetilde{\Gamma}_o & \widetilde{G}_o^t & \varrho\widetilde{A}_o^t\widetilde{\mathcal{W}} \\ 0 & \widetilde{G}_{do}^t & \varrho\widetilde{A}_d^t\widetilde{\mathcal{W}} \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{4.32}
$$

Consider the following partition matrices:

$$
\begin{aligned}
\widetilde{\mathcal{X}} &= \widetilde{P}^{-1} = \begin{bmatrix} \mathcal{X}_s & 0 \\ 0 & \mathcal{X}_s \end{bmatrix},\ \widetilde{\mathcal{Q}} = \begin{bmatrix} \mathcal{Q}_a & 0 \\ 0 & \mathcal{Q}_a \end{bmatrix} \\
\widetilde{\mathcal{W}}_a &= \begin{bmatrix} \widetilde{\mathcal{W}}_{aa} & 0 \\ 0 & \widetilde{\mathcal{W}}_{aa} \end{bmatrix},\ \widetilde{\mathcal{W}}_c = \begin{bmatrix} \widetilde{\mathcal{W}}_{cc} & 0 \\ 0 & \widetilde{\mathcal{W}}_{cc} \end{bmatrix}, \\
\widetilde{\mathcal{R}} &= \begin{bmatrix} \mathcal{R}_a & 0 \\ 0 & \mathcal{R}_a \end{bmatrix}, \widetilde{\mathcal{Z}} = \begin{bmatrix} \mathcal{Z} & 0 \\ 0 & \mathcal{Z} \end{bmatrix},\ \widetilde{\mathcal{G}} = \begin{bmatrix} \mathcal{G} & 0 \\ 0 & \mathcal{G} \end{bmatrix} \\
\widetilde{T} &= diag\begin{bmatrix} I & I & I \end{bmatrix},\ \widehat{T} = diag\begin{bmatrix} \widetilde{\mathcal{X}} & \widetilde{\mathcal{X}} & \widetilde{\mathcal{X}} & \widetilde{\mathcal{X}} \end{bmatrix}
\end{aligned} \tag{4.33}
$$

Thes matrices are helpful in controller design. The following theorems state the main result of dynamic feedback stabilization:

Theorem 4.4 *Given scalars ϱ and μ, the system 4.1 with dynamic output-feedback* *controller 4.28 is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ if there exist symmetric positive definite matrices $\mathcal{X},\ \mathcal{G},\ \mathcal{Z}, \{\Theta\}_1^2$, matrices $\mathcal{Y}_a,\ \mathcal{Y}_c,\ \mathcal{Y}_s,\ \mathcal{Y}_m$, $\{\Theta\}_3^8$ and a scalar $\gamma > 0$ satisfying the following LMI:*

$$\begin{bmatrix} \Sigma_o & \varrho\widetilde{\Upsilon} & \varrho\widehat{\Upsilon} & \Sigma_v \\ \bullet & -\varrho\widetilde{\mathcal{Z}} & 0 & 0 \\ \bullet & \bullet & -\varrho\widetilde{\mathcal{G}} & 0 \\ \bullet & \bullet & \bullet & -\Sigma_w \end{bmatrix} < 0 \tag{4.34}$$

where

$$\begin{aligned}
\Sigma_o &= \begin{bmatrix} \Sigma_{o1} & \Sigma_{o2} & \Sigma_{o3} \\ \bullet & \Sigma_{o4} & \Sigma_{o5} \\ \bullet & \bullet & \Sigma_{o6} \end{bmatrix},\ \widetilde{\Theta}_j = diag\begin{bmatrix} \Theta_j & \Theta_j \end{bmatrix} \\
\Sigma_{o1} &= \begin{bmatrix} \Sigma_{o11} & \Sigma_{o12} \\ \bullet & \Sigma_{o13} \end{bmatrix}, \\
\Sigma_{o11} &= \mathcal{X}_s A_o^t + A_o\mathcal{X}_s + \Theta_1 + \Theta_2 + \Theta_3 + \Theta_3^t + \Theta_6 + \Theta_6^t,\ \Sigma_{o12} = \mathcal{Y}_c^t + B_o\mathcal{Y}_a, \\
\Sigma_{o13} &= \mathcal{X}_s A_o^t + A_o\mathcal{X}_s - \mathcal{Y}_s - \mathcal{Y}_s^t + \Theta_1 + \Theta_2 + \Theta_3 + \Theta_3^t + \Theta_6 + \Theta_6^t, \\
\Sigma_{o2} &= \begin{bmatrix} A_{do}\mathcal{X}_s - 2\Theta_3 + \Theta_4^t + \Theta_7^t & 0 \\ \mathcal{Y}_m & -2\Theta_3 + \Theta_4^t + \Theta_7^t \end{bmatrix}, \\
\Sigma_{o3} &= \begin{bmatrix} \Theta_3 - \Theta_6 + \Theta_8^t + \Theta_5^t & 0 \\ \bullet & \Theta_3 - \Theta_6 + \Theta_8^t + \Theta_5^t \end{bmatrix}, \\
\Sigma_{o4} &= \begin{bmatrix} \Sigma_{o41} & 0 \\ \bullet & \Sigma_{o41} \end{bmatrix},\ \Sigma_{o41} = -(1-\mu)\Theta_1 - 2\Theta_4 - 2\Theta_4^t,
\end{aligned}$$

$$\Sigma_{o5} = \begin{bmatrix} \Theta_4 - \Theta_7 - 2\Theta_5^t & 0 \\ \bullet & \Theta_4 - \Theta_7 - 2\Theta_5^t \end{bmatrix},$$

$$\Sigma_{o6} = \begin{bmatrix} \Sigma_{o61} & 0 \\ \bullet & \Sigma_{o61} \end{bmatrix}, \; \Sigma_{o61} = -\Theta_2 + \Theta_5 + \Theta_5^t - \Theta_8 - \Theta_8^t,$$

$$\widehat{\Upsilon} = \begin{bmatrix} \Theta_3 \\ \Theta_4 \\ \Theta_5 \end{bmatrix}, \; \widetilde{\Upsilon} = \begin{bmatrix} \Theta_6 \\ \Theta_7 \\ \Theta_8 \end{bmatrix}, \; \Sigma_v = \begin{bmatrix} \Gamma_o & \Sigma_c & \varrho\Sigma_a \\ 0 & \Sigma_p & \varrho\Sigma_n \\ 0 & 0 & 0 \end{bmatrix},$$

$$\Sigma_a = \begin{bmatrix} (\mathcal{Z}+\mathcal{G})A_o^t & (\beta_a+\beta_c)\mathcal{Y}_c^t \\ (\beta_a+\beta_c)\mathcal{Y}_a^t & (\mathcal{Z}+\mathcal{G})A_o^t - (\beta_a+\beta_c)\mathcal{Y}_s^t \end{bmatrix}, \; \Sigma_c = \begin{bmatrix} \mathcal{X}_s G_o^t \\ \mathcal{Y}_a^t D_o^t \end{bmatrix},$$

$$\Sigma_n = \begin{bmatrix} (\mathcal{Z}+\mathcal{G})A_{do}^t & (\beta_a+\beta_c)\mathcal{Y}_m^t \\ 0 & 0 \end{bmatrix}, \; \Sigma_p = \begin{bmatrix} \mathcal{X}_s G_{do}^t \\ 0 \end{bmatrix},$$

$$\Sigma_w = \begin{bmatrix} \gamma^2 I & -\Phi_o^t & -\varrho(\mathcal{Z}+\mathcal{G})\Gamma_o \\ \bullet & I & K_o\Psi_o \\ \bullet & \bullet & \varrho(\mathcal{Z}+\mathcal{G}) \end{bmatrix} \tag{4.35}$$

Moreover, the gain matrix is given by $K_c = \mathcal{Y}_a\mathcal{X}_s^{-1}$.

Proof: Introduce $\mathcal{W}_a = \beta_a\mathcal{P},\ \mathcal{W}_c = \beta_c\mathcal{P},\ \beta_a > 0,\ \beta_c > 0$. Apply the congruent transformation $\mathcal{T} = diag[\widehat{T},\ \widetilde{T}]$ to the LMI 4.24 using the linearizations:

$$\begin{aligned}
\mathcal{Y}_a &= K_c\mathcal{X}_s,\ \mathcal{Y}_c = K_oC_o\mathcal{X}_s,\ \mathcal{Y}_s = K_o(C_o - F_oK_c)\mathcal{X}_s + B_o\mathcal{Y}_a, \\
\Theta_1 &= \mathcal{X}\mathcal{Q}\mathcal{X},\ \Theta_2 = \mathcal{X}\mathcal{R}\mathcal{X},\ \Theta_3 = \mathcal{X}N_a\mathcal{X},\ \mathcal{Y}_m = K_oC_{do}\mathcal{X}_s, \\
\Theta_4 &= \mathcal{X}N_c\mathcal{X},\ \Theta_5 = \mathcal{X}N_s\mathcal{X},\ \Theta_6 = \mathcal{X}M_a\mathcal{X}, \\
\Theta_7 &= \mathcal{X}M_c\mathcal{X},\ \Theta_8 = \mathcal{X}M_s\mathcal{X},\ \mathcal{Z} = \beta_a\mathcal{X},\ \mathcal{G} = \beta_c\mathcal{X}
\end{aligned}$$

and the matrix definitions 4.33, we obtain the LMI 4.34 by Schur's complements.

This method can be used in different ways. One way is to find, for a given K_o, the K_c that stabilizes the system. When K_o is given, many terms in the LMI become constants. Another way is to find a K_o for a given K_c by using the following theorem:

Theorem 4.5 *Given scalars ϱ, feedback gain K_c and μ, the system 4.1 with dynamic output-feedback controller (4.28) is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ if there exist symmetric matrices $P,\ P_c,\ ,\mathcal{G},\ \mathcal{Z}$, weighting matrices $\mathcal{Y}_o,\ Q,\ R,\ N_a$,*
$N_c,\ N_s,\ M_a,\ M_c,\ M_s$, and a scalar $\gamma > 0$ satisfying the following LMI:

$$\begin{bmatrix} \Sigma_o & \varrho\widehat{\mathcal{M}} & \varrho\widehat{\mathcal{N}} & \Sigma_v \\ \bullet & -\varrho\widetilde{\mathcal{Z}} & 0 & 0 \\ \bullet & \bullet & -\varrho\widetilde{\mathcal{G}} & 0 \\ \bullet & \bullet & \bullet & -\Sigma_w \end{bmatrix} < 0 \tag{4.36}$$

$$\begin{aligned}
\Sigma_o &= \begin{bmatrix} \Sigma_{o1} & \Sigma_{o2} & \Sigma_{o3} \\ \bullet & \Sigma_{o4} & \Sigma_{o5} \\ \bullet & \bullet & \Sigma_{o6} \end{bmatrix},\ \Sigma_{o1} = \begin{bmatrix} \Sigma_{o11} & \Sigma_{o12} \\ \bullet & \Sigma_{o13} \end{bmatrix}, \\
\Sigma_{o11} &= PA_c + A_c^tP + R + Q + N_a + N_a^t + M_a + M_a^t,\ A_c = A_o + B_oK_c \\
\Sigma_{o12} &= -PBK_c,\ \Sigma_{o13} = P_cA_o + A_o^tP_c - \mathcal{Y}_o(C_o - F_oK_c) - (C_o^t - K_c^tF_o^t)\mathcal{Y}_o^t \\
&+ R + Q + N_a + N_a^t + M_a + M_a^t \\
\Sigma_{o2} &= \begin{bmatrix} PA_{do} - 2N_a + N_c^t + M_c^t & 0 \\ P_cA_{do} - \mathcal{Y}_oC_{do} & -2N_a + N_c^t + M_c^t \end{bmatrix}, \\
\Sigma_{o3} &= \begin{bmatrix} N_a - M_a + N_s^t + M_s^t & 0 \\ \bullet & N_a - M_a + N_s^t + M_s^t \end{bmatrix},\ \Sigma_{o4} = \begin{bmatrix} \Sigma_{o41} & 0 \\ \bullet & \Sigma_{o41} \end{bmatrix}, \\
\Sigma_{o41} &= -(1-\mu)Q - 2N_c - 2N_c^t
\end{aligned}$$

$$\begin{aligned}
\Sigma_{o5} &= \begin{bmatrix} N_c - M_c - 2N_s^t & 0 \\ \bullet & N_c - M_c - 2N_s^t \end{bmatrix}, \; \Sigma_{o6} = \begin{bmatrix} \Sigma_{o61} & 0 \\ \bullet & \Sigma_{o61} \end{bmatrix}, \\
\Sigma_{o61} &= -R + N_s + N_s^t - M_s - M_s^t, \\
\widehat{\mathcal{M}} &= \begin{bmatrix} \widetilde{M}_a \\ \widetilde{M}_c \\ \widetilde{M}_s \end{bmatrix}, \quad \widehat{\mathcal{N}} = \begin{bmatrix} \widetilde{N}_a \\ \widetilde{N}_c \\ \widetilde{N}_s \end{bmatrix} \quad \Sigma_v = \begin{bmatrix} \widetilde{\Gamma}_o & \Sigma_c & \varrho\Sigma_a \\ 0 & \Sigma_p & \varrho\Sigma_n \\ 0 & 0 & 0 \end{bmatrix}, \\
\widetilde{J}_k &= diag[J_k, J_k] \;\; J = M, N, \;\; k = a, c \;\; and \;\; s \\
\widetilde{\mathcal{Z}} &= \begin{bmatrix} \mathcal{Z} & 0 \\ 0 & \beta_a P_c \end{bmatrix}, \; \widetilde{\mathcal{G}} = \begin{bmatrix} \mathcal{G} & 0 \\ 0 & \beta_b P_c \end{bmatrix}, \\
\widetilde{\Gamma}_o &= \begin{bmatrix} P\Gamma_o \\ P_c\Gamma_o - \mathcal{Y}_o\Psi \end{bmatrix}, \quad \Sigma_c = \begin{bmatrix} G_o^t + K_c^t D_o^t \\ -K_c^t D_o^t \end{bmatrix}, \; \Sigma_p = \begin{bmatrix} G_{do}^t \\ 0 \end{bmatrix}, \\
\Sigma_n &= \begin{bmatrix} A_{do}^t(\mathcal{Z}+\mathcal{G}) & (\beta_a+\beta_c)A_{do}^t P_c - (\beta_a+\beta_c)C_{do}\mathcal{Y}_o^t \\ 0 & 0 \end{bmatrix}, \\
\Sigma_a &= \begin{bmatrix} A_c^t(\mathcal{Z}+\mathcal{G}) & 0 \\ K_c^t B^t(\mathcal{Z}+\mathcal{G}) & (\beta_a+\beta_c)A_o^t P_c - (\beta_a+\beta_c)(C_o^t - K_c^t F_o^t)\mathcal{Y}_o^t \end{bmatrix}, \\
\Sigma_w &= \begin{bmatrix} \gamma^2 I & -\Phi_o^t & -\rho\Sigma_r \\ \bullet & I & 0 \\ \bullet & \bullet & \varrho\Sigma_s \end{bmatrix} \; \Sigma_s = \begin{bmatrix} (\mathcal{Z}+\mathcal{G}) & 0 \\ 0 & (\beta_a+\beta_c)P_c \end{bmatrix}, \\
\Sigma_r &= \begin{bmatrix} \Gamma_o^t(\mathcal{Z}+\mathcal{G}) & (\beta_a+\beta_c)\Gamma_o^t P_c - (\beta_a+\beta_c)\Psi_o\mathcal{Y}_o^t \end{bmatrix}
\end{aligned} \tag{4.37}$$

Moreover, the gain matrix is given by $K_o = P_c^{-1}\mathcal{Y}_0$.

Proof: the theorem can be obtained by straightforward substitution in 4.31 for a given K_c by considering the following transformation:

$$\begin{aligned}\dot{\widetilde{x}}(t) &= \widetilde{A}_o x(t) + \widetilde{A}_d x(t-\tau) + \widetilde{\Gamma}_o w(t), \\ \widetilde{z}(t) &= \widetilde{G}_o x(t) + \widetilde{G}_d x(t-\tau) + \Phi_o w(t) \end{aligned} \tag{4.38}$$

where K_o is to be determined and the states and the matrices of the transformed system are given by:

$$\begin{aligned}\widetilde{x} &= \begin{bmatrix} x^t(t) & x^t(t) - x_c{}^t(t) \end{bmatrix}^t \quad \widetilde{G}_d = \begin{bmatrix} G_{do} & 0 \end{bmatrix} \\ \widetilde{A}_o &= \begin{bmatrix} A_o + B_o K_c & -B_o K_c \\ 0 & A_o - K_o C_o \end{bmatrix}, \ \widetilde{\Gamma}_o = \begin{bmatrix} \Gamma_o \\ \Gamma_o - K_o \Psi_o \end{bmatrix} \\ \widetilde{A}_d &= \begin{bmatrix} A_{do} & 0 \\ A_{do} - K_o C_{do} & 0 \end{bmatrix}, \ \widetilde{G}_o = \begin{bmatrix} G_o + D_o K_o & -D_o K_c \end{bmatrix} \end{aligned} \tag{4.39}$$

with

$$\widetilde{\mathcal{P}} = \begin{bmatrix} P & 0 \\ 0 & P_c \end{bmatrix} \quad \widetilde{\mathcal{Z}} = \begin{bmatrix} \mathcal{Z} & 0 \\ 0 & \beta_a P_c \end{bmatrix}, \ \widetilde{\mathcal{G}} = \begin{bmatrix} \mathcal{G} & 0 \\ 0 & \beta_b P_c \end{bmatrix}$$

where β_a and β_b are positive numbers.

By this theorem, one can use the K_c obtained in the state feedback part, and then use this theorem to find K_o.

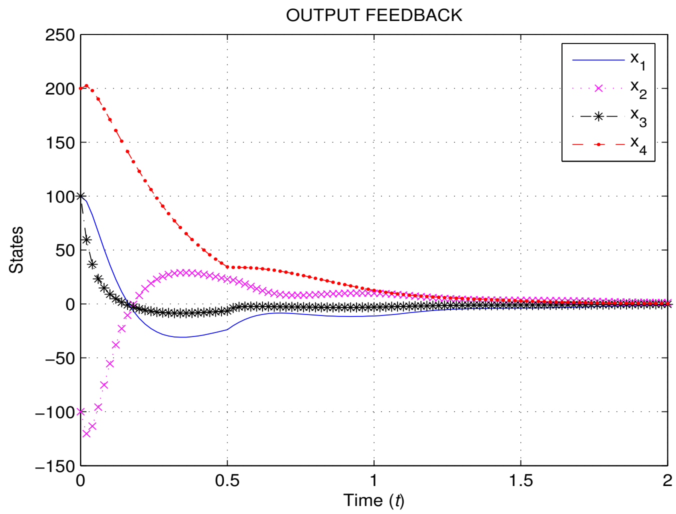

Figure 4.3: State-trajectories by output-feedback: Example 4.3

4.3.3 Example 4.3

Consider the chemical reactor treated in Examples 4.1 and 4.2 with the additional data:

$$C_o = \begin{bmatrix} 1 & 10 & 0 & 0 \\ 0 & 0 & 10 & 10 \end{bmatrix} \quad C_{do} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad F_o^t = [0, 0], \; \Psi^t = [0.2, 0.1]$$

Application of Theorem 4.3 using the solver LMI-Toolbox yields the solution:

$$\begin{aligned}
\mu &= 2, \; \varrho = 0.775, \; \gamma = 0.3149, \; \beta_a = 0.819, \; \beta_c = 0.081 \\
K_s &= \begin{bmatrix} 7.5351 & 1.7396 & -49.5242 & -3.9617 \\ -30.5670 & -10.1238 & 110.4315 & 2.4973 \end{bmatrix}, \\
K_o^t &= \begin{bmatrix} -89.1039 & 188.5843 & -40.0866 & 134.8605 \\ -10.6072 & 22.8563 & -4.8094 & 17.0193 \end{bmatrix}
\end{aligned}$$

The corresponding closed-loop states-responses are plotted in Figure 4.3. For the chemical system considered thus far, the trajectories depicted in Figs 4.1 to 4.3 show that the state-feedback and dynamic output-feedback are quite comparable in regulating the system.

4.4 Conclusions

We provided an efficient solution to the problem of delay-dependent analysis and feedback synthesis for a class of linear continuous-time systems with time-varying delays. We constructed an augmented Lyapunov-Krasovskii functional and deployed an improved free-weighting method to exhibit the delay-dependent dynamics. Delay-dependent stability analysis was subsequently performed to develop conditions in the form of linear matrix inequalities (LMIs) whose feasibility guarantees that the linear delay system is asymptotically stable with a $\gamma-$level $\mathcal{L}_2-$gain. The superiority of the developed method in comparison with the existing methods was established. We designed state-feedback and dynamic output-feedback schemes to guarantee that the closed-loop switched system enjoys the delay-dependent asymptotic stability with a prescribed $\gamma-$level $\mathcal{L}_2-$gain. The established results were extended to systems with convex-bounded parameter uncertainties in all system matrices, and they were then tested on representative examples.

CHAPTER 5

SECOND APPROACH

5.1 Introduction

In this chapter, further improvements are made over the methods developed in the previous chapter. A new class of systems is considered which includes unknown time-varying nonlinearities that satisfy the Lipschitz conditions. An appropriate Lyapunov functional is constructed to exhibit the delay-dependent dynamics. Then, a robust feedback stabilization method is given on the basis of state measurements. A new method to design observer-based output feedback is also introduced. Both types of controllers are designed so as to guarantee that the corresponding closed-loop system enjoys the delay-dependent robust stability with an $\mathcal{L}_2-$gain smaller than a prescribed constant level. For linear systems, and by using simpler LMIs, the method presented in this chapter succeeds in giving similar results to those obtained in Chapter 4. All the developed results are expressed in terms of convex optimization over LMIs and tested on representative examples.

5.2 Problem Statement

The approach adopted in this chapter is based on recasting a general class of nominally-linear time-delay (NLTD) systems as:

$$\begin{aligned}
\dot{x}(t) &= A_o x(t) + A_{do} x(t-\tau) + B_o u(t) + f_o(x(t),t) + h_o(x(t-\tau),t) + \Gamma_o w(t), \\
y(t) &= C_o x(t) + C_{do} x(t-\tau) + \Psi_o w(t), \\
z(t) &= G_o x(t) + G_{do} x(t-\tau) + D_o u(t) + \Phi_o w(t) \qquad (5.1)
\end{aligned}$$

The matrices have the same characteristics as those mentioned in Chapter 4.

In the sequel, it is assumed that the delay $\tau(t)$ is a differentiable time-varying function satisfying $0 < \tau(t) \leq \varrho, \quad \dot{\tau}(t) \leq \mu$ where the bounds ϱ and μ are known constant scalars. The unknown functions $f_o = f_o(x(t),t) \in \Re^n,\ h_o = h_o(x(t),t) \in \Re^n$ are vector-valued time-varying nonlinear perturbations with $f_o(0,t) = 0,\ h_o(0,t) = 0 \forall\ t$ and satisfy the following Lipschitz condition for all $(x,t), (\hat{x},t) \in \Re^n X \Re$:

$$\begin{aligned}
||f_o(x(t),t) \ - \ f_o(\hat{x}(t),t)|| &\leq \alpha\ ||F(x-\hat{x})|| \\
||h_o(x(t-\tau),t) \ - \ h_o(\hat{x}(t-\tau),t)|| &\leq \beta\ ||H\ (x(t-\tau) - \hat{x}(t-\tau))|| \qquad (5.2)
\end{aligned}$$

for some constant $\alpha > 0$ and $\beta > 0$ and $F \in \Re^{n\ n},\ H \in \Re^{n\ n}$ are constant matrices. Note, as a consequence of 5.2, we have:

$$\begin{aligned}
||f_o(x(t),t)|| &\leq \alpha\ ||F\ x|| \\
||h_o(x(t-\tau),t)|| &\leq \beta\ ||H\ x(t-\tau)|| \qquad (5.3)
\end{aligned}$$

Equivalently stated, the condition 5.2 implies:

$$\begin{aligned} &\Big[f_o^t(x(t),t)f_o(x(t),t) \;-\; \alpha^2 x^t(t)F^tFx(t)\Big] \leq 0 \\ &\Big[h_o^t(x(t-\tau),t)h_o(x(t-\tau),t) \;-\; \beta^2 x^t(t-\tau)H^tHx(t-\tau)\Big] \leq 0 \end{aligned} \qquad (5.4)$$

Remark 5.2.1 *It should be observed that the system 5.1 can be easily derived by simple arrangements and algebraic manipulations of some families of nonlinear functions, and not necessarily applying any sort of linearizations. The system has one state delay, but all the analysis in this paper can be carried over to multiple delays in a straightforward way [54], [27], [18]. Therefore, the system 5.1 has a general structure and all the developed results in this paper, by and large, would encompass the existing published results as special cases.*

Our purpose is to develop a robust criterion for delay-dependent asymptotic stability and stabilization of the system 5.1 with a prescribed performance measure. This criterion aims to reduce the design conservatism usually encountered in time-delay systems.

5.3 Delay-Dependent Stability Analysis

In the sequel, we develop a new criterion for LMI-based characterization of delay-dependent asymptotic stability and $\mathcal{L}_2-$gain analysis. The criterion includes some free-weighting matrices in order to expand the range of applicability of the developed conditions. The following theorem establishes the main result for the NLTD system:

Theorem 5.1 *Given $\varrho > 0$ and $\mu > 0$. The system 5.1 with $u(.) \equiv 0$ is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ if there exist symmetric* *matrices* $\mathcal{P} > 0,\ \mathcal{Q} > 0,\ \mathcal{R} > 0,\ \mathcal{W} > 0,$ *weighting matrices* $N_a,\ N_c$ *and scalars* $\gamma > 0,\ \sigma > 0,\ \kappa > 0$ *satisfying:*

$$\Xi = \begin{bmatrix} \Xi_o & \varrho\mathcal{N} & \widehat{\mathcal{P}} & \widehat{\mathcal{P}} & \Xi_x \\ \bullet & -\varrho\mathcal{W} & 0 & 0 & 0 \\ \bullet & \bullet & -\sigma I & 0 & 0 \\ \bullet & \bullet & \bullet & -\kappa I & 0 \\ \bullet & \bullet & \bullet & \bullet & -\Xi_y \end{bmatrix} < 0 \tag{5.5}$$

where

$$\begin{aligned} \Xi_o &= \begin{bmatrix} \Xi_{o1} & \Xi_{o2} & N_a \\ \bullet & \Xi_{o3} & N_c \\ \bullet & \bullet & -\mathcal{R} \end{bmatrix} \ \Xi_{o2} = \mathcal{P}A_{do} - 2N_a + N_c^t, \\ \Xi_{o1} &= \mathcal{P}A_o + A_o^t\mathcal{P} + \mathcal{Q} + \mathcal{R} + N_a + N_a^t + \sigma\alpha^2 F^t F \\ \Xi_{o3} &= -(1-\mu)\mathcal{Q} - 2N_c - 2N_c^t + \kappa\beta^2 H^t H \\ \mathcal{N} &= \begin{bmatrix} N_a \\ N_c \\ 0 \end{bmatrix},\ \Xi_y = \begin{bmatrix} \gamma^2 I & -\Phi_o^t & -\varrho\Gamma_o^t\mathcal{W} \\ \bullet & I & 0 \\ \bullet & \bullet & \varrho\mathcal{W} \end{bmatrix}, \\ \Xi_x &= \begin{bmatrix} \mathcal{P}\Gamma_o & G_o^t & \varrho A_o^t\mathcal{W} \\ 0 & G_{do}^t & \varrho A_{do}^t\mathcal{W} \\ 0 & 0 & 0 \end{bmatrix},\ \widehat{\mathcal{P}} = \begin{bmatrix} \mathcal{P} \\ 0 \\ 0 \end{bmatrix} \end{aligned} \tag{5.6}$$

Proof : Consider the Lyapunov-Krasovskii functional (LKF):

$$V(t) \ = \ V_o(t) + V_m(t) + V_c(t) + V_a(t)$$

where

$$
\begin{aligned}
V_o(t) &= x^t(t)\mathcal{P}x(t), \\
V_a(t) &= \int_{t-\tau(t)}^{t} x^t(s)\mathcal{Q}x(s)\ ds, \\
V_c(t) &= \int_{t-\varrho}^{t} x^t(s)\mathcal{R}x(s)ds, \\
V_m(t) &= \int_{-\varrho}^{t}\int_{t+s}^{t} \dot{x}^t(\alpha)\mathcal{W}\dot{x}(\alpha)d\alpha\ ds \qquad (5.7)
\end{aligned}
$$

where $0 < \mathcal{P} = \mathcal{P}^t,\ 0 < \mathcal{W} = \mathcal{W}^t,\ 0 < \mathcal{Q} = \mathcal{Q}^t,\ 0 < \mathcal{R} = \mathcal{R}^t$ are matrices of appropriate dimensions. A straightforward computation gives the time-derivative of $V(x)$ along the solutions of 5.1 with $w(t) \equiv 0$ as:

$$
\begin{aligned}
\dot{V}_o(t) &= 2x^t\mathcal{P}[A_o x(t) + A_{do}x(t-\tau) + f_o + h_o] \qquad (5.8) \\
\dot{V}_a(t) &= x^t(t)\mathcal{Q}x(t) - (1-\dot{\tau})\ x^t(t-\tau(t))\mathcal{Q}x(t-\tau(t)) \\
&\le x^t(t)\mathcal{Q}x(t) - (1-\mu)\ x^t(t-\tau(t))\mathcal{Q}x(t-\tau(t)) \qquad (5.9) \\
\dot{V}_c(t) &= x^t(t)\mathcal{R}x(t) - x^t(t-\varrho)\mathcal{R}x(t-\varrho) \qquad (5.10) \\
\dot{V}_m(t) &= \varrho\ \dot{x}^t(t)\mathcal{W}\dot{x}(t) - \int_{t-\varrho}^{0} \dot{x}^t(s)\mathcal{W}\dot{x}(s)ds \qquad (5.11)
\end{aligned}
$$

In terms of:

$$
\xi(t) = \begin{bmatrix} x^t(t) & x^t(t-\tau(t)) & x^t(t-\varrho) \end{bmatrix}^t
$$

By using the classical Leibniz rule $x(t-\theta) = x(t) - \int_{t-\theta}^{t} \dot{x}(s)ds$ for any matrices $N_a,\ N_c$ of appropriate dimensions and using $\mathcal{N}$ as in 5.6, the following equations hold:

$$2\,\xi^t(t)(2\mathcal{N})\Big[-\int_{t-\tau(t)}^{t}\dot{x}(s)ds + x(t) - x(t-\tau)\Big] = 0,$$
$$2\,\xi^t(t)(-\mathcal{N})\Big[-\int_{t-\varrho}^{t}\dot{x}(s)ds + x(t) - x(t-\varrho)\Big] = 0 \qquad (5.12)$$

From 5.7-5.11 and using 5.12, we have:

$$\begin{aligned}
\dot{V}(t)|_{5.1} &\leq x^t(t)[\mathcal{P}A_o + A_o^t\mathcal{P} + \mathcal{Q} + \mathcal{R} + N_a + N_a^t]x(t) \\
&+2x^t(t)\mathcal{P}[f_o + h_o] + 2x^t(t)[\mathcal{P}A_{do} - 2N_a + N_c^t]x(t-\tau) \\
&+2x^t(t)N_a x(t-\varrho) + 2x^t(t-\tau)N_c x(t-\varrho) \\
&-x^t(t-\tau)[(1-\mu)\mathcal{Q} + 2N_c + 2N_c^t]x(t-\tau(t)) \\
&-x^t(t-\varrho)\mathcal{R}x(t-\varrho) - 2\,\xi^t(t)(2\mathcal{N})\int_{t-\tau}^{t}\dot{x}(s)ds \\
&-\int_{t-\varrho}^{t}\dot{x}^t(s)\mathcal{W}\dot{x}(s)ds + 2\,\xi^t(t)\mathcal{N}\int_{t-\varrho}^{t}\dot{x}(s)ds \\
&+\varrho\,\xi^t(t)\mathcal{A}_g\mathcal{W}\mathcal{A}_g^t\xi(t) \qquad (5.13) \\
\mathcal{A}_g &= \begin{bmatrix} A_o^t & A_{do}^t & 0 \end{bmatrix}^t
\end{aligned}$$

$\dot{V}(x)|_{5.1}$ defines the Lyapunov derivative along the solutions of the system 5.1. Regrouping terms of 5.13 and taking into account 5.4 via the S-procedure [40], it follows that there exist scalars $\sigma > 0,\ \kappa > 0$ such that the use of manipulating 5.13 and Schur's complements leads to:

$$\begin{aligned}
\dot{V}(t)|_{5.1} &= \xi^t(t)\Xi_o\xi(t) + \sigma f_o^t f_o + \kappa h_o^t h_o + 2x^t\mathcal{P}[f_o + h_o] \\
&- \int_{t-\varrho}^{t}\dot{x}^t(s)\mathcal{W}\dot{x}(s)ds + \xi^t(t)\mathcal{A}_g\mathcal{W}\mathcal{A}_g^t\xi(t) \\
&- 2\xi^t(t)\mathcal{N}\int_{t-\tau(t)}^{t}\dot{x}(s)ds - 2\xi^t(t)(-\mathcal{N})\int_{t-\varrho}^{t-\tau(t)}\dot{x}(s)ds \qquad (5.14)
\end{aligned}$$

where matrices Ξ_o, $\mathcal{N}$ are given in 5.6. Steps Similar to those made in Chapter 4 of 5.14 yield:

$$\begin{aligned}
\dot{V}(t)|_{5.1} &\leq \chi^t(t,s)\,\widehat{\Xi}\,\chi(t,s) \\
&- \int_{t-\tau(t)}^{t} [\xi^t\mathcal{N} + \dot{x}^t\mathcal{W}]\mathcal{W}^{-1}[\xi^t\mathcal{N} + \dot{x}^t\mathcal{W}]^t ds \\
&- \int_{t-\varrho}^{t-\tau(t)} [-\xi^t\mathcal{N} + \dot{x}^t\mathcal{W}]\mathcal{W}^{-1}[-\xi^t\mathcal{N} + \dot{x}^t\mathcal{W}]^t ds \\
&\leq \chi^t(t,s)\,\widehat{\Xi}\,\chi(t,s) \qquad (5.15) \\
&\chi(t,s) = \begin{bmatrix} x^t(t) & x^t(t-\tau(t)) & \dot{x}^t(s) & f_o^t & h_o^t \end{bmatrix}^t \qquad (5.16)
\end{aligned}$$

In view of 5.5 with $G_o \equiv 0$, $G_d \equiv 0$, $\Gamma_o \equiv 0$, and Schur's complements, it follows from 5.15 that $\dot{V}(t)|_{5.1} < 0$ which establishes the internal asymptotic stability.

Consider the performance measure:

$$J = \int_0^\infty \left(z^t(s)z(s) - \gamma^2 w^t(s)w(s) \right) ds$$

For any $w(t) \in \mathcal{L}_2(0,\infty) \neq 0$ and zero initial condition $x(0) = 0$, we have:

$$J \leq \int_0^\infty \left(z^t(s)z(s) - \gamma^2 w^t(s)w(s) + \dot{V}(x)|_{5.1} \right) ds$$

Proceeding as before, we get:

$$\begin{aligned}
&z^t(s)z(s) - \gamma^2 w^t(s)w(s) + \dot{V}(s)|_{5.1} = \bar{\chi}^t(t,s)\,\bar{\Xi}\,\bar{\chi}(t,s), \\
&\bar{\chi}(t,s) = \begin{bmatrix} \bar{\chi}^t(t,s) & w^t(s) \end{bmatrix}^t
\end{aligned}$$

where $\bar{\Xi}$ corresponds to Ξ in 5.5 by Schur's complements. It is readily seen from 5.5 by Schur's complements that:

$$z^t(s)z(s) - \gamma^2 w^t(s)w(s) \ + \dot{V}(s)|_{5.1} < 0$$

for arbitrary $s \in [t, \infty)$, which implies for any $w(t) \in \mathcal{L}_2(0, \infty) \neq 0$ that $J < 0$ leading to $||z(t)||_2 \ < \ \gamma \ ||w(t)||_2$ and the proof is completed.

5.3.1 Nonlinear Uncertain Systems

Suppose now that the system 5.1 has the following state-space model:

$$\begin{aligned} \dot{x}(t) &= A_\Delta x(t) + A_{d\Delta} x(t-\tau) + B_\Delta u(t) + f_\Delta(x(t), t) + h_\Delta(x(t-\tau), t) + \Gamma_\Delta w(t), \\ y(t) &= C_\Delta x(t) + C_{d\Delta} x(t-\tau) + \Psi_\Delta w(t) \\ z(t) &= G_\Delta x(t) + G_{d\Delta} x(t-\tau) + \Phi_\Delta w(t) \end{aligned} \tag{5.17}$$

whose matrices contain uncertainties which belong to a real convex bounded polytypic model of the type:

$$\begin{bmatrix} A_\Delta & A_{d\Delta} & B_\Delta & \Gamma_\Delta \\ C_\Delta & C_{d\Delta} & & \Psi_\Delta \\ G_\Delta & G_{d\Delta} & D_\Delta & \Phi_\Delta \end{bmatrix} \triangleq$$
$$\left\{ \begin{bmatrix} A_{o\lambda} & A_{d\lambda} & B_{o\lambda} & \Gamma_{o\lambda} \\ C_{o\lambda} & C_{d\lambda} & & \Psi_{o\lambda} \\ G_{o\lambda} & G_{d\lambda} & D_{o\lambda} & \Phi_{o\lambda} \end{bmatrix} = \sum_{j=1}^{N} \lambda_j \begin{bmatrix} A_{oj} & A_{dj} & B_{oj} & \Gamma_{oj} \\ C_{oj} & C_{dj} & & \Psi_{oj} \\ G_{oj} & G_{dj} & D_{oj} & \Phi_{oj} \end{bmatrix}, \lambda \in \Lambda \right\} \tag{5.18}$$

where Λ is the unit simplex given by:

$$\Lambda \triangleq \left\{ (\lambda_1, \cdots, \lambda_N) : \sum_{j=1}^{N} \lambda_j = 1 \; , \; \lambda_j \geq 0 \right\}$$

Define the vertex set $\mathcal{N} = \{1, ..., N\}$. We use $\{A_o, ..., \Phi_o\}$ to imply generic system matrices and $\{A_{oj}, ..., \Phi_{oj}, \; j \in \mathcal{N}\}$ to represent the respective values at the vertices. For the nonlinear part of the system 5.1, the unknown functions $f_\Delta, \; h_\Delta$ satisfy the following bounding conditions:

$$\begin{aligned} ||f_\Delta(x(t), t)|| \; &\leq \; \alpha \, ||F \, x||, \\ ||h_\Delta(x(t-\tau), t)|| \; &\leq \; \beta \, ||H \, x(t-\tau)|| \end{aligned} \tag{5.19}$$

Based on these definitions, the stability of uncertain time delay system with polytypic-type uncertainty can be checked by the following theorem:

Theorem 5.2 *Given $\varrho > 0$ and $\mu > 0$. The system 5.1 with $u(.) \equiv 0$ and polytypic representation 5.18 is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ if there exist symmetric matrices $\mathcal{P} > 0, \; \mathcal{Q} > 0, \; \mathcal{R} > 0, \; \mathcal{Z} > 0,$ weighting matrices $N_a, \; N_c$ and scalars $\gamma > 0, \; \sigma > 0, \; \kappa > 0$ satisfying the following LMIs:*

$$\begin{bmatrix} \Xi_{oj} & \varrho\mathcal{N} & \widehat{\mathcal{P}} & \widehat{\mathcal{P}} & \Xi_{xj} \\ \bullet & -\varrho\mathcal{W} & 0 & 0 & 0 \\ \bullet & \bullet & -\sigma I & 0 & 0 \\ \bullet & \bullet & \bullet & -\kappa I & 0 \\ \bullet & \bullet & \bullet & \bullet & -\Xi_{yj} \end{bmatrix} \tag{5.20}$$

where $j = 1, ..., N$ *and:*

$$\begin{aligned}
\Xi_{oj} &= \begin{bmatrix} \Xi_{o1j} & \Xi_{o2j} & N_a \\ \bullet & \Xi_{o3} & N_c \\ \bullet & \bullet & -\mathcal{R} \end{bmatrix}, \ \Xi_{o2j} = \mathcal{P}A_{dj} - 2N_c + N_c^t \\
\Xi_{o1j} &= \mathcal{P}A_{oj} + A_{oj}^t\mathcal{P} + \mathcal{Q} + \mathcal{R} + N_a + N_a^t + \sigma\alpha^2 F^t F, \\
\Xi_{yj} &= \begin{bmatrix} \gamma^2 I & -\Phi_{oj}^t & -\varrho\Gamma_{oj}^t\mathcal{W} \\ \bullet & I & 0 \\ \bullet & \bullet & \varrho\mathcal{W} \end{bmatrix}, \\
\Xi_{xj} &= \begin{bmatrix} \mathcal{P}\Gamma_{oj} & G_{oj}^t & \varrho A_{oj}^t\mathcal{W} \\ 0 & G_{dj}^t & \varrho A_{dj}^t\mathcal{W} \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{5.21}$$

5.4 Reduction to Linear Systems

The results developed in the last section are reduced hereafter to the following linear model:

$$\begin{aligned}
\dot{x}(t) &= A_o x(t) + A_{do} x(t-\tau) + B_o u(t) + \Gamma_o w(t), \\
y(t) &= C_o x(t) + C_{do} x(t-\tau) + \Psi_o w(t) \\
z(t) &= G_o x(t) + G_{do} x(t-\tau) + D_o u(t) + \Phi_o w(t)
\end{aligned} \tag{5.22}$$

The following corollaries stand out:

Corollary 5.4.1 *Given* $\varrho > 0$ *and* $\mu > 0$. *The system 5.1 with* $u(.) \equiv 0$ *is delay-dependent asymptotically stable with* $\mathcal{L}_2$*-performance bound* γ *if there exist symmetric matrices* $\mathcal{P} > 0, \ \mathcal{Q} > 0, \ \mathcal{R} > 0, \ \mathcal{W} > 0,$ *weighting matrices* $N_a, \ N_c$ *and a scalar*

$\gamma > 0$ *satisfying the following LMI:*

$$\Xi = \begin{bmatrix} \tilde{\tilde{\Xi}}_o & \varrho\mathcal{N} & \Xi_x \\ \bullet & -\varrho\mathcal{W} & 0 \\ \bullet & \bullet & -\Xi_y \end{bmatrix} < 0 \tag{5.23}$$

where

$$\begin{aligned} \tilde{\tilde{\Xi}}_o &= \begin{bmatrix} \tilde{\tilde{\Xi}}_{o1} & \Xi_{o2} & N_a \\ \bullet & \tilde{\tilde{\Xi}}_{o3} & N_c \\ \bullet & \bullet & -\mathcal{R} \end{bmatrix}, \\ \tilde{\tilde{\Xi}}_{o3} &= -(1-\mu)\mathcal{Q} - 2N_c - 2N_c^t \\ \tilde{\tilde{\Xi}}_{o1} &= \mathcal{P}A_o + A_o^t\mathcal{P} + \mathcal{Q} + \mathcal{R} + N_a + N_a^t \end{aligned} \tag{5.24}$$

Corollary 5.4.2 *Given $\varrho > 0$ and $\mu > 0$. The system 5.1 with $u(.) \equiv 0$ and polytypic representation 5.18 is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ if there exist symmetric matrices $\mathcal{P} > 0,\ \mathcal{Q} > 0,\ \mathcal{R} > 0,\ \mathcal{W} > 0,$ weighting matrices $N_a,\ N_c$ and a scalar $\gamma > 0$ satisfying the following LMIs for $j = 1, ..., N$:*

$$\Xi_j = \begin{bmatrix} \tilde{\tilde{\Xi}}_{oj} & \varrho\mathcal{N} & \Xi_{xj} \\ \bullet & -\varrho\mathcal{W} & 0 \\ \bullet & \bullet & -\Xi_{yj} \end{bmatrix} < 0 \tag{5.25}$$

$$\begin{aligned} \tilde{\tilde{\Xi}}_{oj} &= \begin{bmatrix} \tilde{\tilde{\Xi}}_{o1j} & \Xi_{o2j} & N_a \\ \bullet & \tilde{\tilde{\Xi}}_{o3} & N_c \\ \bullet & \bullet & -\mathcal{R} \end{bmatrix}, \\ \tilde{\tilde{\Xi}}_{o1j} &= \mathcal{P}A_{oj} + A_{oj}^t\mathcal{P} + \mathcal{Q} + \mathcal{R} + N_a + N_a^t \end{aligned} \tag{5.26}$$

Method	N_v	N_i	T_e	ϱ
Yong 2007 [58]	54	100	14.27 s	0.9
Corollary 5.4.1	20	100	3.77 s	1.1

TABLE 5.1: Second approach Yong 2007 example computational summary ($\mu = 2$)

5.4.1 Example 5.1

Consider the following second-order system which was treated in [58]:

$$A_o = \begin{bmatrix} -0.5 & -2 \\ 1 & -1 \end{bmatrix}, \quad A_{do} = \begin{bmatrix} -0.5 & -1 \\ 0 & -0.6 \end{bmatrix}$$

The method developed in the present chapter succeeds in getting better results than [58]. The method developed in the present chapter uses fewer system variables and requires less excution time than [58]. A sequence of numerical experiments is performed on a standard computing facility (*This is composed of Intel core due- 2.66 G Hz both processors with 980MB RAM employing Matlab 7*). Table 5.1 contains a summary of the computational results of our methods as compared to [58].

The results thereby validate the superiority of our method. First, in terms of getting less conservative results, the method in [58] is able to prove the stability of the system 5.27 up to $\rho = 0.9$ whereas the method developed in the present chapter proves that the system 5.27 is stable up to $\rho = 1.1$ with improvement over 22 percent. Second, in terms of using fewer variables, the method developed in the present chapter uses 20 variables in this example, which is less than half of the number of variables required by the method in [58]. As a consequence of using fewer variables, the method developed in the present chapter takes less computation time than the method in [58]. Also the use of fewer variables helps in extending the method to design different types of controllers, as will be shown later.

Method	N_v	N_i	T_e	ϱ
Yong 2007 [58]	204	10	14.66 s	0.652
Theorem 5.1	72	10	2.295 s	0.874

TABLE 5.2: Second approach reactor example computational summary ($\mu = 2$)

5.4.2 Example 5.2

An open-loop stable time-delay system for the chemical reactor used for the Example 4.2 in Chapter 4 is considered here. The matrices are given here again:

$$\begin{aligned}
A_o &= \begin{bmatrix} -4.93 & -1.01 & 0 & 0 \\ -3.2 & -5.3 & -12.8 & 0 \\ 6.4 & 0.347 & -32.5 & -1.04 \\ 0 & 0.833 & 11.0 & -3.96 \end{bmatrix}, \; \Gamma_o = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}, \\
A_{do} &= \begin{bmatrix} 1.92 & 0 & 0 & 0 \\ 0 & 1.92 & 0 & 0 \\ 0 & 0 & 1.87 & 0 \\ 0 & 0 & 0 & 0.724 \end{bmatrix}, \Phi = 0.1 \\
G_o &= [0.1 \;\; 0.2 \;\; 0.4 \;\; 0.3], \; G_{do} = [0.01 \;\; 0 \;\; 0.01 \;\; 0]
\end{aligned}$$

The open loop system response was shown in Chapter 4 (Figure 4.1). Table 5.2 summarizes the computational results of our method as compared to the method in [58].

It is evidently clear that our method is quite superior to [58] since the computational time of our method is much less, and their storage requirement is almost three times that of our method, which is quite excessive.

5.5 Feedback Stabilization

As Chapter 3 shows, it is helpful to have methods that design controllers to stabilize time delay systems. In the rest of this chapter, attention is directed to feedback stabilization schemes. The method developed in the Theorem 5.1 is going to be extended to the design of a stabilizing controller. For stabilization purposes, Theorem 5.1 is expected to give good results for two reasons. First, it has less conservative results than earlier methods. Second, it uses fewer variables, and so the stabilization LMIs will be less complicated. The system in 5.27 is going to be changed such that its stability cannot be proved by any method, and the simulation shows an unstable response. Then the stabilization method must stabilize this system. Earlier stabilization methods like the one in [60] cannot design a stabilizing controller for a system which is unstable when $\rho = 0$. The remaining methods mentioned in Chapter 3 do not consider a stabilizing controller design.

Different types of controllers can be designed. In the present thesis, we consider state feedback and dynamic output feedback. None of the methods that was mentioned in Chapter 3 uses a dynamic output controller . This type of controller is very important, since in some cases it is impractical or even impossible to get the values of all the states. Therefore an observer is required to estimate the unmeasured states. These estimated states are going to be used for the feedback. We begin with state-feedback in this section, and subsequently we turn to dynamic output feedback.

5.5.1 State-feedback

Applying the state-feedback control $u(t) = K_s x(t)$ to the nonlinear system 5.1 and define $A_s = A_o + B_o K_s$ and $G_s = G_o + D_o K_s$. It then follows from Theorem 5.1 that the resulting closed-loop system is delay-dependent asymptotically stable with

$\mathcal{L}_2$-performance bound γ if there exist symmetric matrices $\mathcal{P} > 0$, $\mathcal{Q} > 0$, $\mathcal{R} > 0$, $\mathcal{W} > 0$, weighting matrices N_a, N_c and scalars $\gamma > 0$, $\sigma > 0$, $\kappa > 0$ satisfying the following LMI:

$$\widehat{\Xi}_s = \begin{bmatrix} \Xi_s & \varrho\mathcal{N} & \mathcal{P} & \mathcal{P} & \Xi_{xs} \\ \bullet & -\varrho\mathcal{W} & 0 & 0 & 0 \\ \bullet & \bullet & -\sigma I & 0 & 0 \\ \bullet & \bullet & \bullet & -\kappa I & 0 \\ \bullet & \bullet & \bullet & \bullet & -\Xi_y \end{bmatrix} < 0 \tag{5.27}$$

where

$$\begin{aligned} \Xi_s &= \begin{bmatrix} \Xi_{o1s} & \Xi_{o2} & N_a \\ \bullet & \Xi_{o3} & N_c \\ \bullet & \bullet & -\mathcal{R} \end{bmatrix} \\ \Xi_{xs} &= \begin{bmatrix} \mathcal{P}\Gamma_o & G_s^t & \varrho A_s^t\mathcal{W} \\ 0 & G_{do}^t & \varrho A_{do}^t\mathcal{W} \\ 0 & 0 & 0 \end{bmatrix} \\ \Xi_{o1s} &= \mathcal{P}A_s + A_s^t\mathcal{P}^t + \mathcal{Q} + \mathcal{R} + N_a + N_a^t + \sigma\alpha^2 F^t F \end{aligned} \tag{5.28}$$

The LMI 5.27 can be used to check how much delay the stability of a closed loop system can be proved. The following theorem can be used to design a state feedback controller:

Theorem 5.3 *Given scalars ϱ, μ. The system 5.1 with $u(t) = K_s x(t)$ is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ if there exist symmetric matrices $\mathcal{X} > 0$, $\mathcal{Q}_a > 0$, $\mathcal{R}_a > 0$, a matrix $\mathcal{Y}$ and weighting matrices Θ_a, Θ_c and* *scalars $\gamma > 0,\ \sigma > 0,\ \kappa > 0$ satisfying the following LMI:*

$$
\Xi = \begin{bmatrix}
\Pi_s & \varrho\Theta & \widehat{\mathcal{I}} & \widehat{\mathcal{I}} & \Pi_x & \bar{F} & \bar{H} \\
\bullet & -\varrho\mathcal{Z}_a & 0 & 0 & 0 & 0 & 0 \\
\bullet & \bullet & -\sigma I & 0 & 0 & 0 & 0 \\
\bullet & \bullet & \bullet & -\kappa I & 0 & 0 & 0 \\
\bullet & \bullet & \bullet & \bullet & -\Pi_y & 0 & 0 \\
\bullet & \bullet & \bullet & \bullet & \bullet & -\sigma I & 0 \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\kappa I
\end{bmatrix} < 0 \tag{5.29}
$$

$$
\begin{aligned}
\Pi_s &= \begin{bmatrix} \Pi_{s1} & \Pi_{s2} & \Theta_a \\ \bullet & \Pi_{s3} & \Theta_c \\ \bullet & \bullet & -\mathcal{R}_a \end{bmatrix}, \quad \mathcal{Z}_c = \alpha_1^{-1}\mathcal{X}, \\
\Pi_{s1} &= \mathcal{X}A_o^t + A_o\mathcal{X} + \mathcal{Q}_a + \mathcal{R}_a + \Theta_a + \Theta_a^t + \mathcal{Y}^t B_o^t + B_o\mathcal{Y},\ \mathcal{Z}_c = 1/\alpha_1\mathcal{X}, \\
\Pi_{s2} &= A_{do}\mathcal{X} - 2\Theta_a + \Theta_c^t,\ \ \Pi_{s3} = -(1-\mu)\mathcal{Q}_a - 2\Theta_c - 2\Theta_c^t,\ \mathcal{Z}_a = \alpha_1\mathcal{X}, \\
\Theta &= \begin{bmatrix} \Theta_a \\ \Theta_c \\ 0 \end{bmatrix}, \ \Pi_y = \begin{bmatrix} \gamma^2 I & -\Phi_o^t & -\varrho\Gamma_o^t \\ \bullet & I & 0 \\ \bullet & \bullet & \varrho\mathcal{Z}_c \end{bmatrix}, \\
\Pi_x &= \begin{bmatrix} \Gamma_o & \mathcal{X}G_o^t & \varrho\mathcal{X}A_o^t + \varrho\mathcal{Y}^t B_o^t \\ 0 & \mathcal{X}G_{do}^t & \varrho\mathcal{X}A_{do}^t \\ 0 & 0 & 0 \end{bmatrix}, \ \widehat{\mathcal{I}} = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}, \\
\bar{F} &= \begin{bmatrix} \sigma\alpha\mathcal{X}F^t \\ 0 \\ 0 \end{bmatrix}, \ \bar{H} = \begin{bmatrix} 0 \\ \kappa\beta\mathcal{X}H^t \\ 0 \end{bmatrix}
\end{aligned}
$$

Moreover, the gain matrix is given by $K_s = \mathcal{Y}\mathcal{X}^{-1}$.

Proof: To design a state feedback controller, first define $\mathcal{X} = \mathcal{P}^{-1}$ and then apply the following congruent transformation $diag[\mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}, I, I, I, I, I]$ to the LMI 5.27 using the linearizations $\mathcal{Y} = K_s\mathcal{X},\ \Theta_a = \mathcal{X}N_a\mathcal{X},\ \mathcal{R}_a = \mathcal{X}\mathcal{R}\mathcal{X}, \mathcal{Q}_a = \mathcal{X}\mathcal{Q}\mathcal{X},\ \mathcal{W} = \alpha_1\mathcal{P}$ we obtain the LMI 5.29 by Schur's complements.

Theorem 5.4 *Given scalars $\varrho,\ \mu$. The system 5.1 with $u(t) = K_s x(t)$ and polytypic representation 5.18 is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ if there exist symmetric matrices $\mathcal{X} > 0,\ \mathcal{Q}_a > 0,\ \mathcal{R}_a > 0$ a matrix $\mathcal{Y}$ and weighting matrices $\Theta_a,\ \Theta_c$ and scalars $\gamma > 0,\ \sigma > 0,\ \kappa > 0$ satisfying the following LMIs for $j = 1, ..., N$:*

$$\Xi_j = \begin{bmatrix} \Pi_{sj} & \varrho\Theta & \widehat{\mathcal{I}} & \widehat{\mathcal{I}} & \Pi_{xj} & \bar{F} & \bar{H} \\ \bullet & -\varrho\mathcal{Z}_a & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & -\sigma I & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\kappa I & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\Pi_{yj} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\sigma I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\kappa I \end{bmatrix} < 0 \tag{5.30}$$

$$\Pi_{sj} = \begin{bmatrix} \Pi_{s1j} & \Pi_{s2j} & \Theta_a \\ \bullet & \Pi_{s3} & \Theta_c \\ \bullet & \bullet & -\mathcal{R}_a \end{bmatrix},\quad \Pi_{yj} = \begin{bmatrix} \gamma^2 I & -\Phi_{oj}^t & -\varrho\Gamma_{oj}^t \\ \bullet & I & 0 \\ \bullet & \bullet & \varrho\mathcal{Z}_c \end{bmatrix},$$

$$\Pi_{s1j} = \mathcal{X}A_{oj}^t + A_{oj}^t\mathcal{X} + \mathcal{Q}_a + \mathcal{R}_a + \Theta_a + \Theta_a^t + \mathcal{Y}^t B_{oj}^t + B_{oj}\mathcal{Y},$$

$$\Pi_{s2j} = A_{dj}\mathcal{X} - 2\Theta_a + \Theta_c^t,$$

$$\Pi_{xj} = \begin{bmatrix} \Gamma_{oj} & \mathcal{X}G_{oj}^t & \varrho\mathcal{X}A_{oj}^t + \varrho\mathcal{Y}^t B_{oj}^t \\ 0 & \mathcal{X}G_{dj}^t & \varrho\mathcal{X}A_{dj}^t \\ 0 & 0 & 0 \end{bmatrix} \tag{5.31}$$

Moreover, the gain matrix is given by $K_s = \mathcal{Y}\mathcal{X}^{-1}$.

Proof: Follows by parallel development to Theorems 5.2 and 5.3.

5.5.2 Example 5.3

Considering the system treated in Example 5.2, the matrix A_{do} is changed slightly to make the system unstable. Direct application of Theorem 5.3 yields the feasible solution as:

$$
\begin{aligned}
A_{do} &= \begin{bmatrix} 2.92 & 0 & 0 & 0 \\ 0 & 2.92 & 0 & 0 \\ 0 & 0 & 2.87 & 0 \\ 0 & 0 & 0 & 2.724 \end{bmatrix}, \\
\varrho &= 0.61,\ \mu = 2,\ \gamma = 0.4752, \\
K_s &= \begin{bmatrix} -34.5094 & -4.9061 & -52.4436 & -130.5408 \\ 17.5392 & -12.7566 & 79.5070 & 157.7008 \end{bmatrix}
\end{aligned}
$$

The unstable open loop response is shown in Figure 5.1 and 5.2 and the closed-loop state response is plotted in Figure 5.3.

5.5.3 Delayed State-feedback

When the upper bound for the delay is known, it can be used in the controller design. An alternative state-feedback scheme is to benefit from the delayed information and apply the delayed state-feedback control law as:

$$
u(t) = K_s x(t) + K_d x(t - \varrho) \tag{5.32}
$$

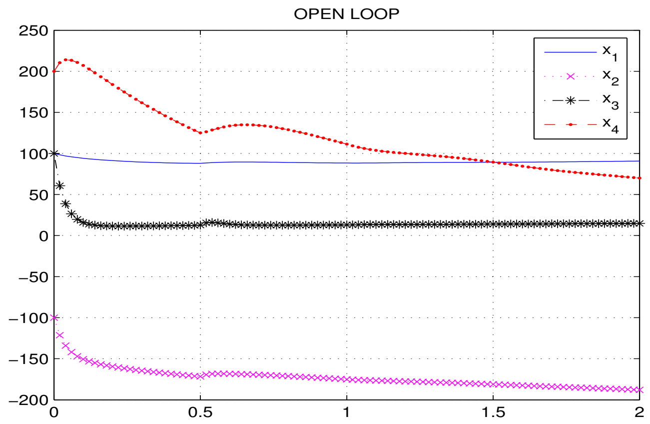

Figure 5.1: Open loop trajectories-Example 5.3

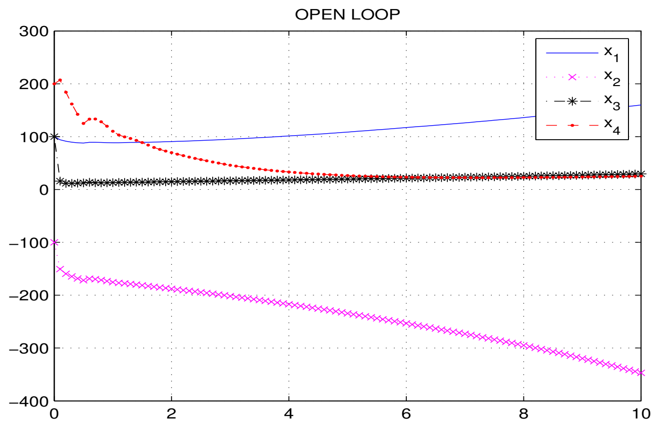

Figure 5.2: Open loop unstable trajectories-Example 5.3

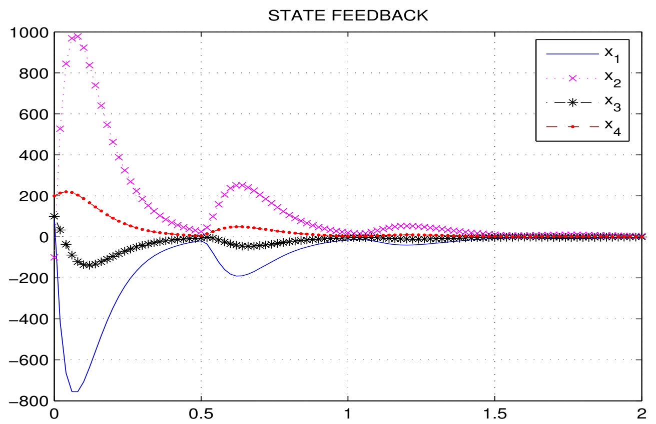

Figure 5.3: State-feedback controlled trajectories-Example 5.3

where $K_s,\ K_d$ are the unknown feedback gains to be determined. Applying the state-feedback control 5.32 to the nonlinear system 5.1 and define $A_s = A_o + B_o K_s$ and $G_s = G_o + D_o K_s$, it then follows from **Theorem 5.1** that the resulting closed-loop system is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ if there exist symmetric matrices $\mathcal{P} > 0,\ \mathcal{Q} > 0,\ \mathcal{R} > 0,\ \mathcal{W} > 0,$ weighting matrices $N_a,\ N_c$ and scalars $\gamma > 0,\ \sigma > 0,\ \kappa > 0$ satisfying the following LMI:

$$\widehat{\Xi}_s = \begin{bmatrix} \Xi_{sd} & \varrho\mathcal{N} & \mathcal{P} & \mathcal{P} & \Xi_{xs} \\ \bullet & -\varrho\mathcal{W} & 0 & 0 & 0 \\ \bullet & \bullet & -\sigma I & 0 & 0 \\ \bullet & \bullet & \bullet & -\kappa I & 0 \\ \bullet & \bullet & \bullet & \bullet & -\Xi_y \end{bmatrix} < 0 \tag{5.33}$$

where

$$\Xi_{sd} = \begin{bmatrix} \Xi_{o1s} & \Xi_{o2} & N_a + \mathcal{P}B_o K_d \\ \bullet & \Xi_{o3} & N_c \\ \bullet & \bullet & -\mathcal{R} \end{bmatrix}, \quad \Xi_{xs} = \begin{bmatrix} \mathcal{P}\Gamma_o & G_s^t & \varrho A_s^t \mathcal{W} \\ 0 & G_{do}^t & \varrho A_{do}^t \mathcal{W} \\ 0 & 0 & \varrho K_d^t B_o^t \mathcal{W} \end{bmatrix} \tag{5.34}$$

Following the foregoing section on state-feedback, the main design results are summarized by the following theorems:

Theorem 5.5 *Given scalars $\varrho,\ \mu$. The system 5.1 with $u(t) = K_s x(t) + K_d x(t-\varrho)$ is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ if there exist symmetric matrices $\mathcal{X} > 0,\ \mathcal{Q}_a > 0,\ \mathcal{R}_a > 0$, matrices $\mathcal{Y}_s,\ \mathcal{Y}_d,$ and weighting* *matrices* $\Theta_a,\ \Theta_c$ *and scalars* $\gamma > 0,\ \sigma > 0,\ \kappa > 0$ *satisfying the following LMI:*

$$\Xi = \begin{bmatrix} \Pi_{sd} & \varrho\Theta & \widehat{\mathcal{I}} & \widehat{\mathcal{I}} & \Pi_x & \bar{F} & \bar{H} \\ \bullet & -\varrho\mathcal{Z}_a & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & -\sigma I & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\kappa I & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\Pi_y & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\sigma I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\kappa I \end{bmatrix} < 0 \tag{5.35}$$

where

$$\begin{aligned} \Pi_{sd} &= \begin{bmatrix} \Pi_{s1} & \Pi_{s2} & \Theta_a + B_o\mathcal{Y}_d \\ \bullet & \Pi_{s3} & \Theta_c \\ \bullet & \bullet & -\mathcal{R}_a \end{bmatrix}, \\ \Pi_{xd} &= \begin{bmatrix} \Gamma_o & \mathcal{X}G_o^t & \varrho\mathcal{X}A_o^t + \varrho\mathcal{Y}_s^t B_o^t \\ 0 & \mathcal{X}G_{do}^t & \varrho\mathcal{X}A_{do}^t \\ 0 & 0 & \varrho\mathcal{Y}_d^t B_o^t \end{bmatrix} \end{aligned} \tag{5.36}$$

and $\Pi_{s1},\ \Pi_{s2}, ..., \Pi_y$ *are given by 5.29. Moreover, the gain matrice for the curresnt values of the states is given by:*

$$K_s = \mathcal{Y}_s\mathcal{X}^{-1}$$

and the gain matrice for the delayed states is:

$$K_d = \mathcal{Y}_d\mathcal{X}^{-1}$$

.

Proof: Using the LKF in 5.7 of the Appendix, we note in this case that:

$$
\begin{aligned}
\dot{V}_o(t) &= 2x^t\mathcal{P}[A_s x(t) + A_{do}x(t-\tau) + f_o + h_o + B_oK_d x(t-\varrho)] && (5.37)\\
\dot{V}_m(t) &= \varrho\, \xi^t(t) \begin{bmatrix} A_o + B_oK_s & A_{do} & B_oK_d \end{bmatrix}^t \mathcal{W} \begin{bmatrix} A_o + B_oK_s & A_{do} & B_oK_d \end{bmatrix} \xi(t)\\
&\quad - \int_{t-\rho}^{t} \dot{x}^t(s)\mathcal{W}\dot{x}(s)ds && (5.38)
\end{aligned}
$$

Now, define $\mathcal{X} = \mathcal{P}^{-1}$ and apply the following congruent transformation to the LMI 5.33:

$$
diag[\mathcal{X}, \mathcal{X}, \mathcal{X}, \mathcal{X}, I, I, I, I, I]
$$

By using the linearizations $\mathcal{Y}_s = K_s\mathcal{X},\ \Theta_a = \mathcal{X}N_a\mathcal{X},\ \mathcal{R}_a = \mathcal{X}\mathcal{R}\mathcal{X},\ \mathcal{Y}_d = K_d\mathcal{X}\ \ \mathcal{Q}_a = \mathcal{X}\mathcal{Q}\mathcal{X},\ \mathcal{W} = \alpha_1\mathcal{P}$ we obtain the LMI 5.35 by Schur's complements.

Theorem 5.6 *Given scalars $\varrho,\ \mu$. The system 5.1 with $u(t) = K_s x(t)$ and polytypic representation 5.18 is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ if there exist symmetric matrices $\mathcal{X} > 0, \mathcal{Q}_a > 0,\ \mathcal{R}_a$ matrices $\mathcal{Y}_s,\ \mathcal{Y}_d$, and weighting matrices $\Theta_a,\ \Theta_c$ and scalars $\gamma > 0,\ \sigma > 0,\ \kappa > 0$ satisfying the following LMIs for $j = 1, ..., N$:*

$$
\Xi_j \quad = \quad \begin{bmatrix}
\Pi_{sdj} & \varrho\Theta & \widehat{\mathcal{I}} & \widehat{\mathcal{I}} & \Pi_{xj} & \bar{F} & \bar{H} \\
\bullet & -\varrho\mathcal{Z}_a & 0 & 0 & 0 & 0 & 0 \\
\bullet & \bullet & -\sigma I & 0 & 0 & 0 & 0 \\
\bullet & \bullet & \bullet & -\kappa I & 0 & 0 & 0 \\
\bullet & \bullet & \bullet & \bullet & -\Pi_{yj} & 0 & 0 \\
\bullet & \bullet & \bullet & \bullet & \bullet & -\sigma I & 0 \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\kappa I
\end{bmatrix} < 0 \qquad (5.39)
$$

where

$$\Pi_{sdj} = \begin{bmatrix} \Pi_{s1j} & \Pi_{s2j} & \Theta_a + B_{oj}\mathcal{Y}_d \\ \bullet & \Pi_{s3} & \Theta_c \\ \bullet & \bullet & -\mathcal{R}_a \end{bmatrix},$$

$$\Pi_{xdj} = \begin{bmatrix} \Gamma_{oj} & \mathcal{X}G_{oj}^t & \varrho\mathcal{X}A_{oj}^t + \varrho\mathcal{Y}^t B_{oj}^t \\ 0 & \mathcal{X}G_{dj}^t & \varrho A_{dj}^t \\ 0 & 0 & \varrho\mathcal{Y}_d^t B_{oj}^t \end{bmatrix} \quad (5.40)$$

Moreover, the gain matrix is given by $K_s = \mathcal{Y}_s\mathcal{X}^{-1},\ K_d = \mathcal{Y}_d\mathcal{X}^{-1}$.

Proof: Follows by parallel development to Theorems 5.2 and 5.3.

5.5.4 Example 5.4

Consider the system treated in Example 5.3 again. Direct application of Theorem 5.5 yields the feasible solution as:

$$\varrho = 0.61,\ \mu = 2,\ \gamma = 0.1832,\ \ \alpha_1 = 0.07$$

$$K_s = \begin{bmatrix} -33.9 & -4.8 & -53.2 & -130.8 \\ 16.7 & -12.9 & 80.3 & 157.8 \end{bmatrix},\ \ K_d = \begin{bmatrix} -1.22 & 0.22 & -3.14 & -8.84 \\ -0.41 & -1.83 & 3.07 & 10.70 \end{bmatrix}$$

5.5.5 Dynamic Output-feedback

In the sequel, we consider stabilizing the system 5.1 by means of the following dynamic output-feedback controller:

$$\begin{aligned} \dot{x}_c(t) &= A_o x_c(t) + B_o u(t) + K_o[y(t) - C_o x_c(t)], \\ u(t) &= K_c x_c(t) \end{aligned} \quad (5.41)$$

Appending the system 5.1 to the controller 5.41, we get the closed-loop nonlinear time-delay (CNTD) system:

$$\begin{aligned}\dot{\widetilde{x}}(t) &= \widetilde{A}_o x(t) + \widetilde{A}_d x(t-\tau) + \widetilde{f}_o(\widetilde{x}(t),t) + \widetilde{h}_o(\widetilde{x}(t-\tau),t) + \widetilde{\Gamma}_o w(t), \\ \widetilde{z}(t) &= \widetilde{G}_o x(t) + \widetilde{G}_d x(t-\tau) + \Phi_o w(t)\end{aligned} \tag{5.42}$$

where K_o and K_c are the unknown gain matrices to be determined and:

$$\begin{aligned}\widetilde{x} &= \begin{bmatrix} x^t(t) & x^t(t) - x_c^t(t) \end{bmatrix}^t = \begin{bmatrix} x^t(t) & e^t(t) \end{bmatrix}^t \\ \widetilde{A}_o &= \begin{bmatrix} A_o + B_o K_c & -B_o K_c \\ 0 & A_o - K_o C_o \end{bmatrix}, \ \widetilde{\Gamma}_o = \begin{bmatrix} \Gamma_o \\ \Gamma_o - K_o \Psi_o \end{bmatrix} \\ \widetilde{A}_d &= \begin{bmatrix} A_{do} & 0 \\ A_{do} - K_o C_{do} & 0 \end{bmatrix}, \ \widetilde{G}_o = \begin{bmatrix} G_o + D_o K_c & -D_o K_c \end{bmatrix} \\ \widetilde{f} &= \begin{bmatrix} f_o \\ f_o \end{bmatrix}, \ \widetilde{h} = \begin{bmatrix} h_o \\ h_o \end{bmatrix}, \ \widetilde{G}_d = \begin{bmatrix} G_{do} & 0 \end{bmatrix}\end{aligned} \tag{5.43}$$

The following two theorems state the main result of dynamic feedback stabilization. In the first theorem, we consider that the state feedback gain K_s and the observer gain K_o are given. The theorem states:

Theorem 5.7 *Given scalars ϱ, μ and matrices $0 < \mathcal{W} = \mathcal{W}^t$. The system 5.1 with dynamic output-feedback controller 5.41 is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ if there exist symmetric matrices $\widetilde{\mathcal{S}} > 0$, $\mathcal{P} > 0$, $\mathcal{Q} > 0$, $\mathcal{R} > 0$, and matrices N_a, N_c, K_c, K_o and scalars $\gamma > 0$, $\sigma > 0$, $\kappa > 0$ satisfying the* *following LMI:*

$$\overline{\Xi}_{df} = \begin{bmatrix} \Pi_{df} & \varrho\widehat{N} & \overline{\mathcal{P}} & \overline{\mathcal{P}} & \Pi_{xf} & \bar{F} & \bar{H} \\ \bullet & -\varrho\mathcal{W} & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & -\sigma I & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\kappa I & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\Pi_{yf} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\sigma I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\kappa I \end{bmatrix} < 0 \tag{5.44}$$

$$\begin{aligned}
\Pi_{df} &= \begin{bmatrix} \Pi_{df1} & \Pi_{df2} & N_a & \Pi_{df4} \\ \bullet & \Pi_{df3} & N_c & \Pi_{df5} \\ \bullet & \bullet & -\mathcal{R} & 0 \\ \bullet & \bullet & \bullet & \Pi_{df6} \end{bmatrix}, \\
\Pi_{df1} &= \mathcal{P}A_o + A_o^t\mathcal{P} + \mathcal{Q} + \mathcal{R} + N_a + N_a^t + \mathcal{P}B_oK_c + K_c^tB_o^t\mathcal{P}, \\
\Pi_{df2} &= \mathcal{P}A_{do} - 2N_a + N_c^t, \ \Pi_{df5} = A_{do}^t\mathcal{S} - C_{do}^tK_o^t\mathcal{S} \\
\Pi_{df3} &= -(1-\mu)\mathcal{Q} - 2N_c - 2N_c^t, \ \Pi_{df4} = -\mathcal{P}B_oK_c, \\
\Pi_{df6} &= \mathcal{S}A_o + A_o^t\mathcal{S} - \mathcal{S}K_oC_o - C_o^tK_o^t\mathcal{S}, \\
\widehat{N} &= \begin{bmatrix} N_a \\ N_c \\ 0 \\ 0 \end{bmatrix}, \ \overline{\mathcal{P}} = \begin{bmatrix} \mathcal{P} \\ 0 \\ 0 \\ \mathcal{S} \end{bmatrix}, \ \ \bar{F} = \begin{bmatrix} \sigma\alpha F \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \bar{H} = \begin{bmatrix} 0 \\ \kappa\beta H \\ 0 \\ 0 \end{bmatrix} \\
\Pi_{xf} &= \begin{bmatrix} \mathcal{P}\Gamma_o & G_o^t + K_c^tD_o^t & \varrho A_o^t\mathcal{W} + \varrho K_c^tB_o^t\mathcal{W} \\ 0 & G_{do}^t & \varrho A_{do}^t\mathcal{W} \\ 0 & 0 & 0 \\ \Pi_{xf1} & \Pi_{xf2} & \Pi_{xf3} \end{bmatrix}, \\
\Pi_{xf1} &= \mathcal{S}\Gamma_o - \mathcal{S}K_o\Psi_o, \ \Pi_{xf2} = -K_c^tD_o, \ \Pi_{xf3} = -\varrho K_c^tB_o^t\mathcal{W}
\end{aligned} \tag{5.45}$$

Proof: Consider the Lyapunov-Krasovskii functional (LKF):

$$\widehat{V}(t) \;=\; V(t) + V_e(t), \quad V_e(t) = e^t(t)\mathcal{S}e(t) \tag{5.46}$$

where $V(t)$ is given by 5.7 and $0 < \mathcal{S} = \mathcal{S}^t$ is a matrix of appropriate dimension. Note from 5.43 that:

$$\begin{aligned}
\dot{V}_o(t) &= 2x^t\mathcal{P}[(A_o + B_oK_c)x(t) + A_{do}x(t-\tau) + f_o + h_o - B_oK_oe(t) + \Gamma_o w(t)] \\
\dot{V}_m(t) &= \varrho\, \dot{x}^t(t)\mathcal{W}\dot{x}(t) - \int_{t-\varrho}^{t} \dot{x}^t(s)\mathcal{W}\dot{x}(s)ds = \varrho\, \eta^t(t)\widehat{\mathcal{W}}\eta(t) - \int_{t-\varrho}^{t} \dot{x}^t(s)\mathcal{W}\dot{x}(s)ds \\
\dot{V}_e(t) &= 2e^t(t)\mathcal{S}[(A_o - K_oC_o)e(t) + (A_{do} - K_oC_{do})x(t-\tau) \\
&+ (\Gamma_o - K_o\Psi_o)w(t) + f_o + h_o],
\end{aligned} \tag{5.47}$$

where

$$\begin{aligned}
\widehat{\mathcal{W}} &= \begin{bmatrix} A_o + B_oK_c & A_{do} & 0 & -B_oK_c \end{bmatrix}^t \mathcal{W} \begin{bmatrix} A_o + B_oK_c & A_{do} & 0 & -B_oK_c \end{bmatrix}, \\
\eta(t) &= \begin{bmatrix} x^t(t) & x^t(t-\tau(t)) & x^t(t-\rho) & e^t(t) \end{bmatrix}^t
\end{aligned}$$

and $\dot{V}_a$, $\dot{V}_c$ remain as before. Proceeding in parallel development to the proof of Theorem 5.1, we conclude from the Lyapunov stability condition $\dot{\widehat{V}}(t) < 0$ that $\overline{\Xi}_{df} < 0$ where $\overline{\Xi}_{df}$ is given by 5.44 subject to 5.45.

This LMI can be used to check the stability of the closed loop time delay system. Based on the principles of separation between the controller and observer gains design, the controller gain K_c can be obtained from theorem 5.3, and then the observer gain K_o can be obtained. In the following theorem, for a given state feedback gain K_c the observer gain K_o can be determined by the following theorem:

Theorem 5.8 *Given scalars ϱ, μ and matrices K_c, $0 < \mathcal{W} = \mathcal{W}^t$. The system 5.1* *with dynamic output-feedback controller 5.41 is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ if there exist symmetric matrices $\mathcal{S} > 0,\ \mathcal{P} > 0,\ \mathcal{Q} > 0,\ \mathcal{R} > 0,$ and matrices $N_a,\ N_c,\ \mathcal{Y}_o$, scalars $\gamma > 0,\ \sigma > 0,\ \kappa > 0$ satisfying the following LMI:*

$$\begin{bmatrix} \Pi_{df} & \varrho\widehat{N} & \overline{\mathcal{P}} & \overline{\mathcal{P}} & \Pi_{xf} & \bar{F} & \bar{H} \\ \bullet & -\varrho\mathcal{W} & 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & -\sigma I & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & -\kappa I & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & -\Pi_{yf} & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\sigma I & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -\kappa I \end{bmatrix} < 0 \tag{5.48}$$

where Π_{df} is given by 5.45 with:

$$\begin{aligned} \Pi_{df1} &= \mathcal{P}A_o + A_o^t\mathcal{P} + \mathcal{Q} + \mathcal{R} + N_a + N_a^t + \mathcal{Y}_c + \mathcal{Y}_c^t, \\ \Pi_{df5} &= A_{do}^t\mathcal{S} - C_{do}^t\mathcal{Y}_o^t,\ \Pi_{df4} = -\mathcal{Y}_c, \\ \Pi_{df6} &= \mathcal{S}A_o + A_o^t\mathcal{S} - \mathcal{Y}_oC_o - C_o^t\mathcal{Y}_o^t, \\ \Pi_{xf} &= \begin{bmatrix} \mathcal{P}\Gamma_o & G_o^t + \mathcal{Y}_s & \varrho A_o^t\mathcal{W} + \varrho\mathcal{Y}_e^t \\ 0 & G_{do}^t & \varrho A_{do}^t\mathcal{W} \\ 0 & 0 & 0 \\ \Pi_{xf1} & \Pi_{xf2} & \Pi_{xf3} \end{bmatrix}, \\ \Pi_{xf1} &= \mathcal{S}\Gamma_o - \mathcal{Y}_o\Psi_o,\ \Pi_{xf2} = -\mathcal{Y}_s,\ \Pi_{xf3} = -\varrho\mathcal{Y}_e^t \\ \mathcal{Y}_s &= D_oK_c,\ \ \mathcal{Y}_c = \mathcal{P}B_oK_c,\ \ \mathcal{Y}_e = \mathcal{W}B_oK_c \end{aligned} \tag{5.49}$$

Moreover, the gain matrix is given by $K_o = \mathcal{S}^{-1}\mathcal{Y}_o$.

Proof: Follows from the proof of Theorem 5.7 along with the linearizations:

$$\mathcal{Y}_o = \mathcal{S}K_o,\ \mathcal{Y}_c = \mathcal{P}B_oK_c,\ \mathcal{Y}_s = D_oK_c,\ ,\ \mathcal{Y}_e = \mathcal{W}B_oK_c$$

5.5.6 Example 5.5

Consider the system in Example 5.3 in addition to the following coefficients:

$$\begin{aligned} C_o &= \begin{bmatrix} 1 & 10 & 0 & 0 \\ 0 & 0 & 10 & 10 \end{bmatrix},\ \Psi_o = \begin{bmatrix} 0.01 & 0.01 \end{bmatrix}, \\ C_d &= \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix},\ D_o = \begin{bmatrix} 0.4 & 0.2 \end{bmatrix} \end{aligned}$$

Direct application of Theorem 5.7 yields the feasible solution as:

$$\begin{aligned} \varrho &= 2.054,\ \mu = 1.12, \\ K_o &= \begin{bmatrix} -217.99 & -20.28 \\ 225.14 & 26.57 \\ -51.05 & -2.79 \\ -27.27 & 0.49 \end{bmatrix}, \\ for & \\ K_c &= \begin{bmatrix} -34.5094 & -4.9061 & -52.4436 & -130.5408 \\ 17.5392 & -12.7566 & 79.5070 & 157.7008 \end{bmatrix} \end{aligned}$$

The closed-loop state response is plotted in Figure 5.4.

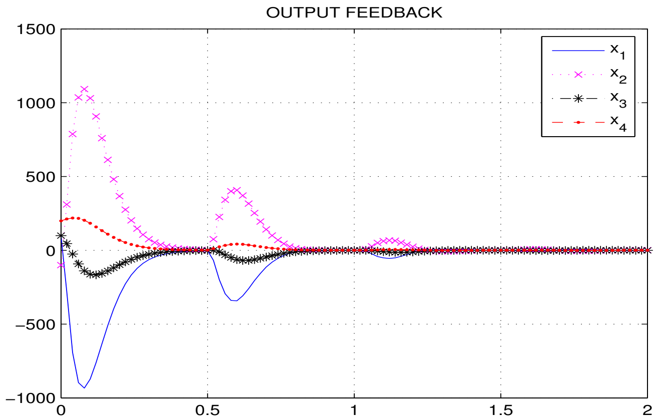

Figure 5.4: Dynamic output feedback-state trajectories-Example 5

5.6 Conclusions

The methods developed in Chapter 4 were simplified in the present chapter. These simplified methods were extended to cover unknown time-varying perturbations satisfying Lipschitz conditions. Steps similar to those made in Chapter 4 were followed, and the developed methods had fewer variables while giving the same results as obtained in Chapter 4. An appropriate Lyapunov functional was constructed to exhibit the delay-dependent dynamics via descriptor format. Delay-dependent stability analysis was performed to characterize conditions based on linear matrix inequalities (LMIs) under which the nonlinear time-delay system is robustly asymptotically stable with a $\gamma-$level $\mathcal{L}_2-$gain. We designed two delay-dependent feedback stabilization schemes: a static one based on state-measurements similar to the one in Chapter 4, and a dynamic one based on observer-based output feedback based on a new LKF. In both schemes, the closed-loop feedback system was shown to enjoy the delay-dependent asymptotic stability with a prescribed $\gamma-$level $\mathcal{L}_2-$gain. The feedback gains were determined by convex optimization over LMIs. All the developed results were tested on representative examples.

CHAPTER 6

THIRD APPROACH

This chapter covers systems with delay of an interval type. Delay of an interval types has both upper and lower limits. This type of delay is very useful for many reasons. First, it is a more general type than the delay type considered in the previous chapters, because if the lower limit is equated to zero, the results for the delay type considered in Chapter 4 and 5 will be obtained. Second, as the present chapter shows, for a given system, longer delay can be tolerated without losing the stability when considering interval delay type. In many time delay systems the delay may vary between two limits. For example, in networked control systems, there is a delay duo to the transmission and access time and this delay cannot be zero, and therefore it has a lower limit. As the data cannot remain in the network forever, there will be an upper limit for the delay.

In this chapter, an LKF is constructed to check the system stability for a delay that has both upper and lower limits. Procedures are developed to design state feedback and dynamic output feedback controllers that will ensure system stability and robustness. All the developed results guarantee that the corresponding linear system enjoys the delay-dependent robust stability with an $\mathcal{L}_2-$gain smaller than a prescribed constant level. The developed theorems are expressed in terms of convex optimization over LMIs and they are tested on representative examples.

6.1 Problem Statement

The following class of the linear nominal time-delay (LNTD) system is considered in the present chapter:

$$\begin{aligned} \dot{x}(t) &= A_o x(t) + A_{do} x(t-\tau) + B_o u(t) + \Gamma_o w(t) \\ y(t) &= C_o x(t) + C_{do} x(t-\tau) + F_o u(t) + \Psi_o w(t) \\ z(t) &= G_o x(t) + G_{do} x(t-\tau) + D_o u(t) + \Phi_o w(t) \end{aligned} \tag{6.1}$$

The matrices have the same dynamic characteristics as those used in the previous chapters. In the following sub-section, it is assumed that the delay $\tau(t)$ is a differentiable time-varying function satisfying:

$$h_l \le \tau(t) \le h_u, \quad \dot{\tau}(t) \le \mu \tag{6.2}$$

where the bounds h_l, h_u and μ are known constant scalars.

Our purpose is to develop robust criteria for delay-dependent asymptotic stability and stabilization of the system 6.1 with a prescribed performance measure.

6.2 Delay-Dependent $\mathcal{L}_2$ Gain Analysis

In this section, a theorem is introduced to check the stability for a given system with time delay of the type given in 6.2:

Theorem 6.1 *Given $h_u > h_l \geq 0$ and $\mu > 0$. The system 6.1 with $u(.) \equiv 0$ is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ if there exist symmetric matrices $0 < \mathcal{P},\ 0 < \mathcal{W}_a,\ 0 < \mathcal{W}_c,\ 0 < \mathcal{Q},\ 0 < \mathcal{R}_1,\ 0 < \mathcal{R}_2$ weighting matrices $N_a,\ N_c, S_a,\ S_c,$ and a scalar $\gamma > 0$ satisfying the following LMI:*

$$\Xi = \begin{bmatrix} \Xi_o & h_u\mathcal{N} & h_d\mathcal{S} & \Xi_x \\ \bullet & -h_u\mathcal{W}_a & 0 & 0 \\ \bullet & \bullet & -h_d\mathcal{W}_c & 0 \\ \bullet & \bullet & \bullet & -\Xi_y \end{bmatrix} < 0 \tag{6.3}$$

where

$$\begin{aligned}
\Xi_o &= \begin{bmatrix} \Xi_{o1} & \Xi_{o2} & N_a + S_a & S_a \\ \bullet & \Xi_{o3} & N_c + S_c & S_c \\ \bullet & \bullet & -\mathcal{R}_1 & 0 \\ \bullet & \bullet & \bullet & -\mathcal{R}_2 \end{bmatrix} \ \mathcal{N} = \begin{bmatrix} N_a \\ N_c \\ 0 \\ 0 \end{bmatrix},\ \mathcal{S} = \begin{bmatrix} S_a \\ S_c \\ 0 \\ 0 \end{bmatrix}, \\
\Xi_{o1} &= \mathcal{P}A_o + A_o^t\mathcal{P} + \mathcal{Q} + \mathcal{R}_1 + \mathcal{R}_2 + N_a + N_a^t \\
\Xi_{o2} &= \mathcal{P}A_{do} - 2N_a - 2S_a + N_c^t, \\
\Xi_{o3} &= -(1-\mu)\mathcal{Q} - 2N_c - 2N_c^t - 2S_c - 2S_c^t, \\
\Xi_y &= \begin{bmatrix} \gamma^2 I & -\Phi_o^t & -\Gamma_o^t(h_u\mathcal{W}_a + h_d\mathcal{W}_c) \\ \bullet & I & 0 \\ \bullet & \bullet & h_u\mathcal{W}_a + h_d\mathcal{W}_c \end{bmatrix}, \\
\Xi_x &= \begin{bmatrix} \mathcal{P}\Gamma_o & G_o^t & A_o^t(h_u\mathcal{W}_a + h_d\mathcal{W}_c) \\ 0 & G_{do}^t & A_{do}^t(h_u\mathcal{W}_a + h_d\mathcal{W}_c) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
& h_d = h_u - h_l
\end{aligned} \tag{6.4}$$

Proof : Define the following variable:

$$\xi(t) = \begin{bmatrix} x^t(t) & x^t(t-\tau(t)) & x^t(t-h_u) & x^t(t-h_l) \end{bmatrix}^t$$

and then by using the classical Leibniz rule $x(t-\theta) = x(t) - \int_{t-\theta}^{t} \dot{x}(s)ds$ for any matrices $N_a,\ N_c,\ S_a,\ S_c,$ of appropriate dimensions, the following equations hold:

$$\begin{aligned}
&2\ \xi^t(t)2\mathcal{N}\left[-\int_{t-\tau(t)}^{t} \dot{x}(s)ds + x(t) - x(t-\tau) \right] = 0 \\
&2\ \xi^t(t)(-\mathcal{N})\left[-\int_{t-h_u}^{t} \dot{x}(s)ds + x(t) - x(t-h_u) \right] = 0 \\
&2\ \xi^t(t)2\mathcal{S}\left[-\int_{t-\tau(t)}^{t-h_l} \dot{x}(s)ds + x(t-h_l) - x(t-\tau) \right] = 0 \\
&2\ \xi^t(t)(-\mathcal{S})\left[-\int_{t-h_u}^{t-h_l} \dot{x}(s)ds + x(t-h_l) - x(t-h_u) \right] = 0
\end{aligned} \tag{6.5}$$

After expanding these terms, we get:

$$\begin{aligned}
&x^t(t)[N_a + N_a^t]x(t) + 2x^t(t)[-2N_a + N_c^t - 2S_a]x(t-\tau(t)) \\
&+2x^t(t)[N_a + S_a]x(t-h_u) + 2x^t(t)[S_a]x(t-h_l) \\
&+2x^t(t-\tau(t))[-2N_c - 2N_c^t - 2S_c - 2S_c^t]x(t-\tau(t)) \\
&+2x^t(t-\tau(t))[N_c + S_c]x(t-h_u) + 2x^t(t-\tau(t))[S_c]x(t-h_l) \\
&-2\ \xi^t(t)2\mathcal{N}\int_{t-\tau(t)}^{t} \dot{x}(s)ds - 2\ \xi^t(t)(-\mathcal{N})\int_{t-h_u}^{t} \dot{x}(s)ds \\
&-2\ \xi^t(t)2\mathcal{S}\int_{t-\tau(t)}^{t-h_l} \dot{x}(s)ds - 2\ \xi^t(t)(-\mathcal{S})\int_{t-h_u}^{t-h_l} \dot{x}(s)ds
\end{aligned} \tag{6.6}$$

Consider now the augmented Lyapunov-Krasovskii functional (ALKF):

$$
\begin{aligned}
V(t) &= V_o(t) + V_{c1}(t) + V_{c2}(t) + V_m(t) + V_{a1}(t) + V_{a2}(t) \\
V_o(t) &= x^t(t)\mathcal{P}x(t), \\
V_{c1}(t) &= \int_{t-h_u}^{t} x^t(s)\mathcal{R}_1 x(s)ds, \\
V_{c2}(t) &= \int_{t-h_l}^{t} x^t(s)\mathcal{R}_2 x(s)ds, \\
V_m(t) &= \int_{t-\tau(t)}^{t} x^t(s)\mathcal{Q}x(s)\ ds, \\
V_{a1}(t) &= \int_{-h_u}^{0}\int_{t+s}^{t} \dot{x}^t(\alpha)(\mathcal{W}_a)\dot{x}(\alpha)d\alpha\ ds, \\
V_{a2}(t) &= \int_{-h_u}^{-h_l}\int_{t+s}^{t} \dot{x}^t(\alpha)(\mathcal{W}_c)\dot{x}(\alpha)d\alpha\ ds \qquad (6.7)
\end{aligned}
$$

where $0 < \mathcal{P} = \mathcal{P}^t,\ 0 < \mathcal{W}_a = \mathcal{W}_a^t,\ 0 < \mathcal{W}_c = \mathcal{W}_c^t,\ 0 < \mathcal{Q} = \mathcal{Q}^t,\ 0 < \mathcal{R}_1 = \mathcal{R}_1^t,$ $0 < \mathcal{R}_2 = \mathcal{R}_2^t$ are matrices of appropriate dimensions. A straightforward computations gives the time-derivative of $V(x)$ along the solutions of 6.1 with $w(t) \equiv 0$ as:

$$
\begin{aligned}
\dot{V}_o(t) &= 2x^t(t)\mathcal{P}\dot{x}(t) = 2x^t(t)\mathcal{P}[A_o x(t) + A_{do}x(t-\tau)] \qquad (6.8) \\
\dot{V}_{a1}(t) &= h_u\ \dot{x}^t(t)\mathcal{W}_a\dot{x}(t) - \int_{t-h_u}^{t} \dot{x}^t(s)\mathcal{W}_a\dot{x}(s)ds \qquad (6.9) \\
\dot{V}_{a2}(t) &= (h_u - h_l)\ \dot{x}^t(t)\mathcal{W}_c\dot{x}(t) - \int_{t-h_u}^{t-h_l} \dot{x}^t(s)\mathcal{W}_c\dot{x}(s)ds \qquad (6.10) \\
\dot{V}_{c1}(t) &= x^t(t)\mathcal{R}_1 x(t) - x^t(t-h_u)\mathcal{R}_1 x(t-h_u) \qquad (6.11) \\
\dot{V}_{c2}(t) &= x^t(t)\mathcal{R}_2 x(t) - x^t(t-h_l)\mathcal{R}_2 x(t-h_l) \qquad (6.12) \\
\dot{V}_m(t) &= x^t(t)\mathcal{Q}x(t) - (1-\dot{\tau})\ x^t(t-\tau(t))\mathcal{Q}x(t-\tau(t)) \\
&\leq x^t(t)\mathcal{Q}x(t) - (1-\mu)\ x^t(t-\tau(t))\mathcal{Q}x(t-\tau(t)) \qquad (6.13)
\end{aligned}
$$

Finally, from 6.7-6.13 and using 6.6, we have:

$$
\begin{aligned}
\dot{V}(t)|_{6.1} \;\leq\;& x^t(t)[\mathcal{P}A_o + A_o^t\mathcal{P} + \mathcal{Q} + \mathcal{R}_1 + \mathcal{R}_2 + N_a + N_a^t]x(t) \\
+\;& 2x^t(t)[\mathcal{P}A_{do} - 2N_a + N_c^t - 2S_a]x(t-\tau) + 2x^t(t)[N_a + S_a]x(t-h_u) \\
+\;& 2x^t(t)[S_a]x(t-h_l) - x^t(t-\tau)[(1-\mu)\mathcal{Q} + 2N_c + 2N_c^t + 2S_c + 2S_c^t]x(t-\tau(t)) \\
+\;& x^t(t-\tau(t))[N_c + S_c]x(t-h_u) + 2x^t(t-\tau(t))[S_c]x(t-h_l) \\
-\;& x^t(t-h_u)\mathcal{R}_1 x^t(t-h_u) - x^t(t-h_l)\mathcal{R}_2 x^t(t-h_l) + h_u\, \dot{x}^t(t)\mathcal{W}_a\dot{x}(t) \\
+\;& h_d\, \dot{x}^t(t)\mathcal{W}_c\dot{x}(t) - \int_{t-h_u}^{t} \dot{x}^t(s)\mathcal{W}_a\dot{x}(s)ds - \int_{t-h_u}^{t-h_l} \dot{x}^t(s)\mathcal{W}_c\dot{x}(s)ds \\
-\;& 2\,\xi^t(t)2\mathcal{N}\int_{t-\tau(t)}^{t} \dot{x}(s)ds - 2\,\xi^t(t)(-\mathcal{N})\int_{t-h_u}^{t} \dot{x}(s)ds \\
-\;& 2\,\xi^t(t)2\mathcal{S}\int_{t-\tau(t)}^{t-h_l} \dot{x}(s)ds - 2\,\xi^t(t)(-\mathcal{S})\int_{t-h_u}^{t-h_l} \dot{x}(s)ds
\end{aligned}
\qquad (6.14)
$$

To manipulate 6.14, first consider:

$$
\begin{aligned}
-\;& 2\,\xi^t(t)2\mathcal{N}\int_{t-\tau(t)}^{t} \dot{x}(s)ds + 2\,\xi^t(t)\mathcal{N}\int_{t-h_u}^{t} \dot{x}(s)ds \\
=\;& -2\,\xi^t(t)2\mathcal{N}\int_{t-\tau(t)}^{t} \dot{x}(s)ds + 2\,\xi^t(t)\mathcal{N}\int_{t-\tau(t)}^{t} \dot{x} + 2\,\xi^t(t)\mathcal{N}\int_{t-h_u}^{t-\tau(t)} \dot{x}(s)ds \\
=\;& -2\,\xi^t(t)\mathcal{N}\int_{t-\tau(t)}^{t} \dot{x}(s)ds + 2\,\xi^t(t)\mathcal{N}\int_{t-h_u}^{t-\tau(t)} \dot{x}(s)ds
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
-\;& 2\,\xi^t(t)2\mathcal{S}\int_{t-\tau(t)}^{t-h_l} \dot{x}(s)ds + 2\,\xi^t(t)\mathcal{S}\int_{t-h_u}^{t-h_l} \dot{x}(s)ds \\
=\;& -2\,\xi^t(t)2\mathcal{S}\int_{t-\tau(t)}^{t-h_l} \dot{x}(s)ds + 2\,\xi^t(t)\mathcal{S}\int_{t-\tau(t)}^{t-h_l} \dot{x} + 2\,\xi^t(t)\mathcal{S}\int_{t-h_u}^{t-\tau(t)} \dot{x}(s)ds \\
=\;& -2\,\xi^t(t)\mathcal{S}\int_{t-\tau(t)}^{t-h_l} \dot{x}(s)ds + 2\,\xi^t(t)\mathcal{S}\int_{t-h_u}^{t-\tau(t)} \dot{x}(s)ds
\end{aligned}
$$

Consider now:

$$
\begin{aligned}
- & \int_{t-h_u}^{t} \dot{x}^t(s)\mathcal{W}_a\dot{x}(s)ds - \int_{t-h_u}^{t-h_l} \dot{x}^t(s)\mathcal{W}_c\dot{x}(s)ds \\
- & \ 2\,\xi^t(t)2\mathcal{N}\int_{t-\tau(t)}^{t} \dot{x}(s)ds - 2\,\xi^t(t)(-\mathcal{N})\int_{t-h_u}^{t} \dot{x}(s)ds \\
- & \ 2\,\xi^t(t)2\mathcal{S}\int_{t-\tau(t)}^{t-h_l} \dot{x}(s)ds - 2\,\xi^t(t)(-\mathcal{S})\int_{t-h_u}^{t-h_l} \dot{x}(s)ds \\
= & -\int_{t-\tau(t)}^{t} \dot{x}^t(s)\mathcal{W}_a\dot{x}(s)ds - \int_{t-h_u}^{t-\tau(t)} \dot{x}^t(s)\mathcal{W}_a\dot{x}(s)ds \\
- & \int_{t-\tau(t)}^{t-h_l} \dot{x}^t(s)\mathcal{W}_c\dot{x}(s)ds - \int_{t-h_u}^{t-\tau(t)} \dot{x}^t(s)\mathcal{W}_c\dot{x}(s)ds \\
- & \ 2\,\xi^t(t)\mathcal{N}\int_{t-\tau(t)}^{t} \dot{x}(s)ds + 2\,\xi^t(t)\mathcal{N}\int_{t-h_u}^{t-\tau(t)} \dot{x}(s)ds \\
- & \ 2\,\xi^t(t)\mathcal{S}\int_{t-\tau(t)}^{t-h_l} \dot{x}(s)ds + 2\,\xi^t(t)\mathcal{S}\int_{t-h_u}^{t-\tau(t)} \dot{x}(s)ds
\end{aligned}
$$

By adding and subtracting the terms:

$$
h_u\xi^t(t)\mathcal{N}\mathcal{W}_a^{-1}\mathcal{N}^t\xi(t) + h_d\xi^t(t)\mathcal{S}\mathcal{W}_c^{-1}\mathcal{S}^t\xi(t)
$$

we find:

$$
\begin{aligned}
& h_u\xi^t(t)\mathcal{N}\mathcal{W}_a^{-1}\mathcal{N}^t\xi(t) + h_d\xi^t(t)\mathcal{S}\mathcal{W}_c^{-1}\mathcal{S}^t\xi(t) \\
& -\int_{t-\tau(t)}^{t} [\xi^t\mathcal{N} + \dot{x}^t(s)\mathcal{W}_a]\mathcal{W}_a^{-1}[\xi^t\mathcal{N} + \dot{x}^t(s)\mathcal{W}_c]^t ds \\
& -\int_{t-h_u}^{t-\tau(t)} [-\xi^t\mathcal{N} + \dot{x}^t(s)\mathcal{W}_a]\mathcal{W}_a^{-1}[-\xi^t\mathcal{N} + \dot{x}^t(s)\mathcal{W}_c]^t ds \\
& -\int_{t-\tau(t)}^{t-h_l} [\xi^t\mathcal{S} + \dot{x}^t(s)\mathcal{W}_c]\mathcal{W}_c^{-1}[\xi^t\mathcal{S} + \dot{x}^t(s)\mathcal{W}_a]^t ds \\
& -\int_{t-h_u}^{t-\tau(t)} [-\xi^t\mathcal{S} + \dot{x}^t(s)\mathcal{W}_c]\mathcal{W}_c^{-1}[-\xi^t\mathcal{S} + \dot{x}^t(s)\mathcal{W}_a]^t ds
\end{aligned}
$$

The last four terms are negative semi-definite, which can be ignored. Further manipulations of 6.15 yield:

$$\begin{aligned}
\dot{V}(t)|_{6.1} &\leq \xi^t(t)\Xi_o\xi(t) - \xi^t(t)\begin{bmatrix} A_o^t \\ A_{do}^t \\ 0 \\ 0 \end{bmatrix}(h_u\mathcal{W}_a + h_d\mathcal{W}_c)\begin{bmatrix} A_o^t \\ A_{do}^t \\ 0 \\ 0 \end{bmatrix}^t \xi(t) \\
&+ h_u\xi^t(t)\mathcal{N}\mathcal{W}_a^{-1}\mathcal{N}^t\xi(t) + h_d\xi^t(t)\mathcal{S}\mathcal{W}_c^{-1}\mathcal{S}^t\xi(t) \qquad (6.15)
\end{aligned}$$

where Ξ_o, $\mathcal{N}$, $\mathcal{S}$ are given in 6.3. In view of 6.3 with $G_o \equiv 0$, $G_d \equiv 0$, $\Gamma_o \equiv 0$, and Schur's complements, it follows that $\dot{V}(t)|_{6.1} < 0$ which establishes the internal asymptotic stability.

Consider the performance measure:

$$J = \int_0^\infty \left(z^t(s)z(s) - \gamma^2 w^t(s)w(s) \right) ds$$

For any $w(t) \in \mathcal{L}_2(0,\infty) \neq 0$ and zero initial condition $x(0) = 0$, we have:

$$\begin{aligned}
J &= \int_0^\infty \left(z^t(s)z(s) - \gamma^2 w^t(s)w(s) + \dot{V}(x)|_{6.1} \right) ds - \dot{V}(x)|_{6.1} \\
&\leq \int_0^\infty \left(z^t(s)z(s) - \gamma^2 w^t(s)w(s) + \dot{V}(x)|_{6.1} \right) ds
\end{aligned}$$

Proceeding as before, we get:

$$\begin{aligned}
& z^t(s)z(s) - \gamma^2 w^t(s)w(s) + \dot{V}(s)|_{6.1} = \bar{\chi}^t(s)\,\bar{\Xi}\,\bar{\chi}(s), \\
& \bar{\chi}(s) = \begin{bmatrix} x^t(s) & x^t(s-\tau(t)) & x^t(t-h_u) & x^t(t-h_l) & w(s) \end{bmatrix}^t \qquad (6.16)
\end{aligned}$$

where $\bar{\Xi}$ corresponds to Ξ in 6.3 by Schur's complements. It is readily seen from 6.3 that:

$$z^t(s)z(s) - \gamma^2 w^t(s)w(s) \ + \dot{V}(s)|_{6.1} < 0$$

for arbitrary $s \in [t, \infty)$, which implies for any $w(t) \in \mathcal{L}_2(0, \infty) \neq 0$ that $J < 0$ leading to $||z(t)||_2 \ < \ \gamma \ ||w(t)||_2$ and the proof is completed.

6.2.1 Linear Uncertain System

Suppose now that the system 6.1 has the following state-space model:

$$\begin{aligned}
\dot{x}(t) &= A_\Delta x(t) + A_{d\Delta} x(t-\tau) + B_\Delta u(t) + \Gamma_\Delta w(t) \\
y(t) &= C_\Delta x(t) + C_{d\Delta} x(t-\tau) + F_\Delta u(t) + \Psi_\Delta w(t) \\
z(t) &= G_\Delta x(t) + G_{d\Delta} x(t-\tau) + D_\Delta u(t) + \Phi_\Delta w(t)
\end{aligned} \tag{6.17}$$

whose matrices contain uncertainties which belong to a real convex bounded polytypic model of the type:

$$\begin{bmatrix} A_\Delta & A_{d\Delta} & B_\Delta & \Gamma_\Delta \\ C_\Delta & C_{d\Delta} & F_\Delta & \Psi_\Delta \\ G_\Delta & G_{d\Delta} & D_\Delta & \Phi_\Delta \end{bmatrix} \in \ \Pi_\lambda$$

$$\triangleq \left\{ \begin{bmatrix} A_\lambda & A_{d\lambda} & B_\lambda & \Gamma_\lambda \\ C_\lambda & C_{d\lambda} & F_\lambda & \Psi_\lambda \\ G_\lambda & G_{d\lambda} & D_\lambda & \Phi_\lambda \end{bmatrix} = \sum_{j=1}^{N} \lambda_j \begin{bmatrix} A_j & A_{dj} & B_j & \Gamma_j \\ C_j & C_{dj} & F_j & \Psi_j \\ G_j & G_{dj} & D_j & \Phi_j \end{bmatrix}, \lambda_j \in \Lambda \right\} \tag{6.18}$$

where Λ is the unit simplex,

$$\Lambda \triangleq \left\{(\lambda_1, \cdots, \lambda_N) : \sum_{j=1}^{N} \lambda_j = 1 \ , \ \lambda_j \geq 0\right\} \tag{6.19}$$

Define the vertex set $\mathcal{N} = \{1, ..., N\}$. We use $\{A_o, ..., \Phi_o\}$ to imply generic system matrices and $\{A_{oj}, ..., \Phi_{oj}, \ j \in \mathcal{N}\}$ to represent the respective values at the vertices. It is a straightforward task to show that the following result holds.

Theorem 6.2 *The system 6.1 with $u(.) \equiv 0$ and polytypic representation 6.18)-6.19 is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ if there exist symmetric matrices $\mathcal{P} > 0, \ \mathcal{Q} > 0, \ \mathcal{R}_1 > 0, \ \mathcal{R}_2 > 0, \ \mathcal{W}_a > 0, \ \mathcal{W}_c > 0,$ weighting matrices $N_a, \ N_c, \ S_a, \ S_c,$ and a scalar γ satisfying the following LMIs for $j = 1, ..., N$:*

$$\Xi_j \ = \ \begin{bmatrix} \Xi_{oj} & h_u\mathcal{N} & h_d\mathcal{S} & \Xi_{xj} \\ \bullet & -h_u\mathcal{W}_a & 0 & 0 \\ \bullet & \bullet & -h_d\mathcal{W}_c & 0 \\ \bullet & \bullet & \bullet & -\Xi_{yj} \end{bmatrix} < 0$$

where

$$\begin{aligned} \Xi_{oj} \ &= \ \begin{bmatrix} \Xi_{o1j} & \Xi_{o2j} & N_a + S_a & S_a \\ \bullet & \Xi_{o3} & N_c + S_c & S_c \\ \bullet & \bullet & -\mathcal{R}_1 & 0 \\ \bullet & \bullet & \bullet & -\mathcal{R}_2 \end{bmatrix} \\ \Xi_{o1} \ &= \ \mathcal{P}A_{oj} + A_{oj}^t\mathcal{P} + \mathcal{Q} + \mathcal{R}_1 + \mathcal{R}_2 + N_a + N_a^t \\ \Xi_{o2} \ &= \ \mathcal{P}A_{doj} - 2N_a - 2S_a + N_c^t, \end{aligned}$$

$$\Xi_{yj} = \begin{bmatrix} \gamma^2 I & -\Phi & \\ {}^{t}_{oj} & -\Gamma^t_{oj}(h_u\mathcal{W}_a + h_d\mathcal{W}_c) & \\ \bullet & I & 0 \\ \bullet & \bullet & h_u\mathcal{W}_a + h_d\mathcal{W}_c \end{bmatrix},$$

$$\Xi_{xj} = \begin{bmatrix} \mathcal{P}\Gamma_{oj} & G^t_{oj} & A^t_{oj}(h_u\mathcal{W}_a + h_d\mathcal{W}_c) \\ 0 & G^t_{doj} & A^t_{doj}(h_u\mathcal{W}_a + h_d\mathcal{W}_c) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (6.20)$$

and $\Xi_{o3},\ \Xi_{o4},\ \Xi_{o5},\ \Xi_{o6}$ *are given in 6.4.*

Remark 6.2.1 *The developed method uses less number of variable than the method proposed by Yong et al. in [59], and also gives less conservative results.*

6.2.2 Example 6.1

The example used by Yong et al. in [59] has the following matrices:

$$A_o = \begin{bmatrix} -2 & 0 \\ 0 & -.9 \end{bmatrix},$$
$$A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

In this example, every time a set of values are put for μ and h_l, then the maximum h_u above which the system fails to satisfy the condition of Theorm 6.1 is reported. Table6.1 shows a comparison of the results obtained by the method in [59], [50] and the those obtained by Theorem 6.1. It is clear that our method has the least conservative results. Furthermore, it uses fewer variables and takes less execustion time. One

h_l	$Method$	$\mu = 0.5$	$\mu = 0.9$
$h_1 = 0$	[50]	1.01	1.01
$h_1 = 0$	[59]	2.04	1.37
$h_1 = 0$	Proposed	2.33	1.87
$h_1 = 2$	[50]	2.39	2.39
$h_1 = 2$	[59]	2.43	2.43
$h_1 = 2$	Proposed	2.6	2.6
$h_1 = 4$	[50]	4.06	4.06
$h_1 = 4$	[59]	4.07	4.07
$h_1 = 4$	Proposed	4.09	4.09

TABLE 6.1: Interval delay type stability results comparison

additional important point is that when h_l is set to zero, the obtained results is exactly what is given by the methods developed in Chapter 4 and Chapter 5. But still the method in Chapter 5 uses fewer variables, the thing that add some advantage to the method in Chapter 5.

6.3 State-Feedback Stabilization

Considering state feedback controller, it is required to apply the state-feedback control $u(t) = K_s x(t)$ to the nominal system 6.1. Define $A_s = A_o + B_o K_s$ and $G_s = G_o + D_o K_s$. It then follows from Theorem 6.1 that the resulting closed-loop system is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ if there exist symmetric matrices $0 < \mathcal{P}$, $0 < \mathcal{W}_a$, $0 < \mathcal{W}_c$, $0 < \mathcal{Q}$, $0 < \mathcal{R}_1$, $0 < \mathcal{R}_2$, weighting matrices N_a, N_c, S_a, S_c, and a scalar $\gamma > 0$ satisfying the following LMI:

$$\Xi_s = \begin{bmatrix} \Xi_{os} & h_u \mathcal{N} & h_d \mathcal{S} & \Xi_{xs} \\ \bullet & -h_u \mathcal{W}_a & 0 & 0 \\ \bullet & \bullet & -h_d \mathcal{W}_c & 0 \\ \bullet & \bullet & \bullet & -\Xi_y \end{bmatrix} < 0 \tag{6.21}$$

where

$$
\begin{aligned}
\Xi_{os} &= \begin{bmatrix} \Xi_{o1s} & \Xi_{o2} & N_a + S_a & S_a \\ \bullet & \Xi_{o3} & N_c + S_c & S_c \\ \bullet & \bullet & -\mathcal{R}_1 & 0 \\ \bullet & \bullet & \bullet & -\mathcal{R}_2 \end{bmatrix} \\
\Xi_{o1} &= \mathcal{P}A_s + A_s^t\mathcal{P} + \mathcal{Q} + \mathcal{R}_1 + \mathcal{R}_2 + N_a + N_a^t \\
\Xi_{o2} &= \mathcal{P}A_{do} - 2N_a - 2S_a + N_c^t, \\
\Xi_{xs} &= \begin{bmatrix} \mathcal{P}\Gamma_{oj} & G_s^t & A_{os}^t(h_u\mathcal{W}_a + h_d\mathcal{W}_c) \\ 0 & G_{do}^t & A_{do}^t(h_u\mathcal{W}_a + h_d\mathcal{W}_c) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\end{aligned} \tag{6.22}
$$

This theorem can be used to check the stability of a closed loop system. To design a controller gain the following theorem can be applied:

Theorem 6.3 *Given scalars ϱ and μ, the system 6.1 with $u(t) = K_s x(t)$ is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ if there exist symmetric matrices $\mathcal{X} > 0$, $\Theta_1^3 > 0$ and matrices $\mathcal{Y}$, Θ_4^7 and a scalar $\gamma > 0$ satisfying the following LMI:*

$$
\begin{bmatrix} \Pi_o & h_u\widehat{\Theta} & h_d\widetilde{\Theta} & \Pi_v \\ \bullet & -h_u\mathcal{Z} & 0 & 0 \\ \bullet & \bullet & -h_d\mathcal{G} & 0 \\ \bullet & \bullet & \bullet & -\Pi_w \end{bmatrix} < 0 \tag{6.23}
$$

where

$$\begin{aligned}
\Pi_o &= \begin{bmatrix} \Pi_{o1} & \Pi_{o2} & \Theta_4+\Theta_6 & \Theta_6 \\ \bullet & \Pi_{o3} & \Theta_5+\Theta_7 & \Theta_7 \\ \bullet & \bullet & -\Theta_2 & 0 \\ \bullet & \bullet & \bullet & -\Theta_3 \end{bmatrix} \\
\Pi_{o1} &= A_o\mathcal{X} + \mathcal{X}A_o^t + B_o\mathcal{Y} + \mathcal{Y}^tB_o^t + \Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 + \Theta_4^t \\
\Pi_{o2} &= A_{do}\mathcal{X} - 2\Theta_4 - 2\Theta_6 + \Theta_5^t, \\
\Pi_{o3} &= -(1-\mu)\Theta_1 - 2\Theta_5 - 2\Theta_5^t - 2\Theta_7 - 2\Theta_7^t, \\
\Pi_v &= \begin{bmatrix} \Gamma_o & \Pi_c & \Pi_a \\ 0 & \mathcal{X}G_{do}^t & \mathcal{X}A_{do}^t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
\Pi_c &= \mathcal{X}G_o^t + \mathcal{Y}^tD_o^t \\
\Pi_a &= \mathcal{X}A_o^t + \mathcal{Y}^tB_o^t \\
\Pi_w &= \begin{bmatrix} \gamma^2 I & -\Phi_o^t & -\Gamma_o^t \\ \bullet & I & 0 \\ \bullet & \bullet & \beta_c\mathcal{X} \end{bmatrix} \\
\beta_c &= \frac{1}{h_u\beta_a + h_d\beta_b}
\end{aligned} \tag{6.24}$$

Moreover, the gain matrix is given by $K_s = \mathcal{Y}\mathcal{X}^{-1}$. β_a and β_c are any positive number

Proof: Introduce $\mathcal{W}_a = \beta_a\mathcal{P}$, $\mathcal{W}_c = \beta_c\mathcal{P}$, $\beta_a > 0$, $\beta_c > 0$. Define $\mathcal{X} = \mathcal{P}^{-1}$ and apply

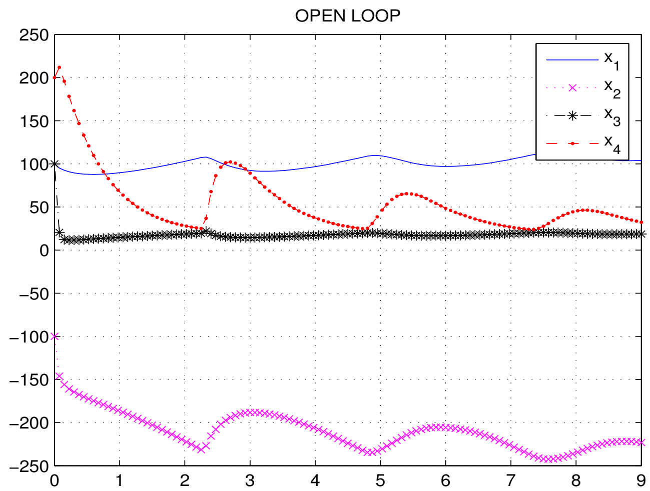

Figure 6.1: Open loop response: Example 6.2

the following congruent transformation:

$$\begin{aligned}
\mathcal{T} &= diag[T_1,\ T_2],\\
T_1 &= diag\begin{bmatrix} \mathcal{X} & \mathcal{X} & \mathcal{X} & \mathcal{X} & \mathcal{X} & \mathcal{X} \end{bmatrix}\\
T_2 &= diag\begin{bmatrix} I & I & I \end{bmatrix}
\end{aligned}$$

to the the LMI 6.21, and by using the linearizations:

$$\begin{aligned}
\mathcal{Z} &= \beta_a\mathcal{X},\ \mathcal{G}_c = \beta_c\mathcal{X},\ \Theta_1 = \mathcal{X}\mathcal{Q}\mathcal{X},\\
\Theta_2 &= \mathcal{X}\mathcal{R}_1\mathcal{X},\ \Theta_3 = \mathcal{X}\mathcal{R}_2\mathcal{X},\ \Theta_4 = \mathcal{X}N_a\mathcal{X},\\
\Theta_5 &= \mathcal{X}N_c\mathcal{X},\ \Theta_6 = \mathcal{X}S_a\mathcal{X},\ \Theta_7 = \mathcal{X}S_c\mathcal{X}
\end{aligned}$$

and the matrix definitions 6.24, we obtain LMI 6.23 by Schur's complements.

6.3.1 Example 6.2

We consider the chemical reactor used in Chapter 5 for Example 5.3:

$$
\begin{aligned}
A_o &= \begin{bmatrix} -4.93 & -1.01 & 0 & 0 \\ -3.2 & -5.3 & -12.8 & 0 \\ 6.4 & 0.347 & -32.5 & -1.04 \\ 0 & 0.833 & 11.0 & -3.96 \end{bmatrix}, \ \Gamma_o = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}, \\
A_{do} &= \begin{bmatrix} 2.92 & 0 & 0 & 0 \\ 0 & 2.92 & 0 & 0 \\ 0 & 0 & 2.87 & 0 \\ 0 & 0 & 0 & 2.724 \end{bmatrix}, \Phi = 0.1 \\
G_o &= [0.1 \ \ 0.2 \ \ 0.4 \ \ 0.3], \ G_{do} = [0.01 \ \ 0 \ \ 0.01 \ \ 0]
\end{aligned}
$$

The delay is assumed to be of the range type with $h_u = 2.5, \quad h_l = 2 \quad \mu = 2$. The open loop system is found to be unstable see Figure 6.1 Then by applying **Theorem 6.3** with:

$$
B_o^t = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},
$$

The found solution is:

$$
\begin{aligned}
&\beta_a = 10^{-5} \ \ \beta_c = 0.07 \ \ \gamma = 0.0198 \\
&K_s = \begin{bmatrix} -40.14 & -5.97 & -70.87 & -178.70 \\ 20.99 & -14.77 & 102.01 & 221.32 \end{bmatrix}
\end{aligned}
$$

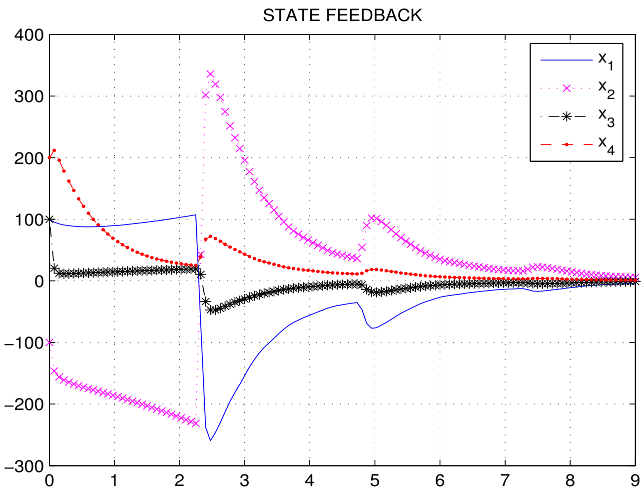

Figure 6.2: Example 6.2 State feedback response

The closed loop response is shown in Figure 6.2. It is clear that the system becomes stable in addition a $\gamma-$level $\mathcal{L}_2-$gain from the disturbance to the controlled output is preserved. The delay effect appears after around 2 seconds, as shown in the figure, but the system succeeds in preserving the stability. The size of the delay here is larger than the one considered in Chapter 5, but because the delay has a limited range a controller can be designed to get an acceptable response.

6.3.2 Dynamic Output-feedback

Here a similar method to the one proposed in Chapter 5 is adopted to design a dynamic output feedback controller. In the sequel, we consider stabilizing the system 6.1 by means of the following dynamic output-feedback controller:

$$\begin{aligned} \dot{x}_c(t) &= A_o x_c(t) + B_o u(t) + K_o[y(t) - C_o x_c(t)], \\ u(t) &= K_c x_c(t) \end{aligned} \tag{6.25}$$

Appending the system 6.1 to controller 6.25, we get the closed-loop time-delay system:

$$\begin{aligned} \dot{\widetilde{x}}(t) &= \widetilde{A}_o x(t) + \widetilde{A}_d x(t-\tau) + \widetilde{\Gamma}_o w(t), \\ \widetilde{z}(t) &= \widetilde{G}_o x(t) + \widetilde{G}_d x(t-\tau) + \Phi_o w(t) \end{aligned} \tag{6.26}$$

where K_o and K_c are the unknown gain matrices to be determined and:

$$\begin{aligned} \widetilde{x} &= \begin{bmatrix} x^t(t) & x^t(t) - x_c^t(t) \end{bmatrix}^t \quad \widetilde{G}_d = \begin{bmatrix} G_{do} & 0 \end{bmatrix} \\ \widetilde{A}_o &= \begin{bmatrix} A_o + B_o K_c & -B_o K_c \\ 0 & A_o - K_o C_o \end{bmatrix}, \ \widetilde{\Gamma}_o = \begin{bmatrix} \Gamma_o \\ \Gamma_o - K_o \Psi_o \end{bmatrix} \\ \widetilde{A}_d &= \begin{bmatrix} A_{do} & 0 \\ A_{do} - K_o C_{do} & 0 \end{bmatrix}, \ \widetilde{G}_o = \begin{bmatrix} G_o + D_o K_c & -D_o K_c \end{bmatrix} \end{aligned} \tag{6.27}$$

where K_o and K_c are the unknown gain matrices to be determined. This method uses fewer variables, and so it needs less execution time. In the following theorem, for a pre-calculated K_c, the gain matrix K_o that stabilizes the system and ensure a $\gamma-$level $\mathcal{L}_2-$gain from the disturbance to the controlled is to be determined.

Theorem 6.4 *Given scalars $h_u > h_l \geq 0,\ \mu$. The system 6.1 with dynamic output-feedback controller 6.26 is delay-dependent asymptotically stable with $\mathcal{L}_2$-performance bound γ if there exist symmetric matrices $\mathcal{P} > 0, \mathcal{Q} > 0, \mathcal{R}_1 > 0, \mathcal{R}_2 > 0, \mathcal{W}_a > 0, \mathcal{W}_b > 0$ and $\mathcal{T} > 0$, matrices $\mathcal{Y}_o, \mathcal{N}_a, \mathcal{N}_c, \mathcal{S}_a$ and $\mathcal{S}_c$ and scalar $\gamma > 0$ satisfying:*

$$\Xi = \begin{bmatrix} \Xi_o & h_u \mathcal{N} & h_d \mathcal{S} & \Xi_x \\ \bullet & -h_u \mathcal{W}_a & 0 & 0 \\ \bullet & \bullet & -h_d \mathcal{W}_c & 0 \\ \bullet & \bullet & \bullet & -\Xi_y \end{bmatrix} < 0$$

$$
\begin{aligned}
\Xi_o &= \begin{bmatrix} \Xi_{o1} & \Xi_{o2} & N_a + S_a & S_a & -PB_oK_c \\ \bullet & \Xi_{o3} & N_c + S_c & S_c & A_d^tT - C_{do}^t\mathcal{Y}_o \\ \bullet & \bullet & -\mathcal{R}_1 & 0 & 0 \\ \bullet & \bullet & \bullet & -\mathcal{R}_2 & 0 \\ \bullet & \bullet & \bullet & \bullet & TA_o + A_o^tT - \mathcal{Y}_oC_o - C_o^t\mathcal{Y}_o^t \end{bmatrix} \\
\Xi_{o1} &= \mathcal{P}A_s + A_s^t\mathcal{P} + \mathcal{Q} + \mathcal{R}_1 + \mathcal{R}_2 + N_a + N_a^t \\
\Xi_{o2} &= \mathcal{P}A_{do} - 2N_a - 2S_a + N_c^t, \\
\Xi_{o3} &= -(1-\mu)\mathcal{Q} - 2N_c - 2N_c^t - 2S_c - 2S_c^t, \\
\mathcal{N} &= \begin{bmatrix} N_a \\ N_c \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \mathcal{S} = \begin{bmatrix} S_a \\ S_c \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\
\Xi_y &= \begin{bmatrix} \gamma^2 I & -\Phi_o^t & -\Gamma_o^t(h_u\mathcal{W}_a + h_d\mathcal{W}_c) \\ \bullet & I & 0 \\ \bullet & \bullet & h_u\mathcal{W}_a + h_d\mathcal{W}_c \end{bmatrix}, \\
\Xi_x &= \begin{bmatrix} \mathcal{P}\Gamma_o & G_o^t + Kc^tD_o^t & A_s^t(h_u\mathcal{W}_a + h_d\mathcal{W}_c) \\ 0 & G_{do}^t & A_{do}^t(h_u\mathcal{W}_a + h_d\mathcal{W}_c) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ T\Gamma_o - \mathcal{Y}_o\Psi & -Kc^tD_o^t & -K_c^tB^t(h_u\mathcal{W}_a + h_d\mathcal{W}_c) \end{bmatrix},
\end{aligned} \tag{6.28}
$$

where $A_s = A_o + B_oK_c,\ h_d = h_u - h_l$ and the observer gain K_o is given by $T^{-1}\mathcal{Y}_o$.

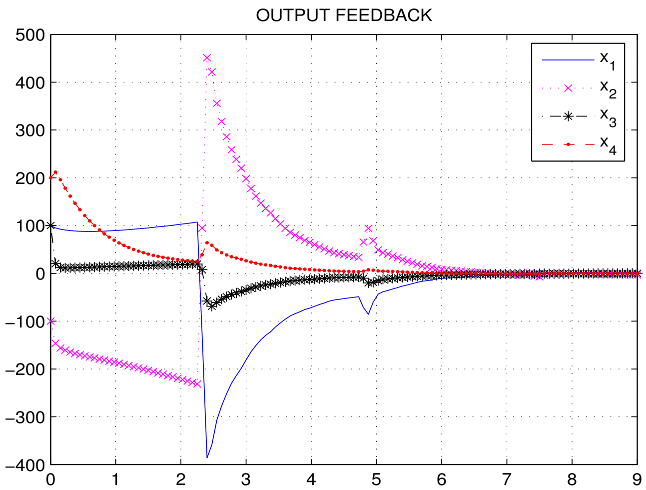

Figure 6.3: Dynamic output feedback response: Example 6.3

6.3.3 Example 6.3

Consider again the system in Example 6.2 with the matrices:

$$
\begin{aligned}
C_o &= \begin{bmatrix} 1 & 10 & 0 & 0 \\ 0 & 0 & 10 & 10 \end{bmatrix}, \; C_d = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}, \\
D_o &= \begin{bmatrix} 0.4 & 0.2 \end{bmatrix} \quad \Psi_o = \begin{bmatrix} 0.01 & 0.01 \end{bmatrix},
\end{aligned}
$$

By using the K_c got from the state feedback theorem, K_o is found to be:

$$
K_o = \begin{bmatrix} -79.32 & -38.67 \\ 92.35 & 47.68 \\ -17.02 & -7.32 \\ -4.9 & -0.59 \end{bmatrix},
$$

The obtained response is shown in Figure 6.3.

From the figure it is clear that the controller succeeded in stabilizing the system and the response is close to that obtained by state feedback, which verifies the theorem.

6.4 Conclusion

New robust delay-dependent stability and stabilization methods for systems with an interval type delay were established in this chapter. An appropriate Lyapunov functional was constructed to exhibit the delay-dependent dynamics via descriptor format. Delay-dependent stability analysis were performed to characterize conditions in the form of linear matrix inequalities (LMIs) under which the interval time-delay system is robustly asymptotically stable with a $\gamma-$level $\mathcal{L}_2-$gain. Two feedback stabilization schemes were designed: a static one based on state-measurements and a dynamic one based on observer-based output feedback. In both schemes, the closed-loop feedback system was shown to enjoy the delay-dependent asymptotic stability. The feedback gains were computed by convex optimization over LMIs. All the developed results were tested on representative examples.

CHAPTER 7

CONCLUSIONS AND FUTURE WORK

This thesis has investigated delay-dependent stability analysis and feedback stabilization problems of time delay systems. It aimed to develop appropriate mathematical tools for classes of linear and nonlinear continuous-time systems with time-varying delays. Delay-dependent stability analysis has been subsequently performed to develop linear matrix inequalities (LMIs)-based conditions the feasibility of which guarantees that the linear delay system is asymptotically stable with a $\gamma-$level $\mathcal{L}_2$-gain. The established results have been extended to systems with convex-bounded parameter uncertainties in all system matrices. The main vehicle has been the constructive use of Lyapunov-Krasovskii method. The main contributions of this thesis can be broadly summarized as follows:

1. For linear continuous-time systems, we have provided an efficient solution to the problems of delay-dependent stability analysis and feedback synthesis by properly constructing an augmented Lyapunov-Krasovskii functional and deploying an improved free-weighting method to exhibit the delay-dependent dy-

namics. The superiority of the developed method in comparison with previous methods has been established. Design theorems for state-feedback and dynamic output-feedback controllers have also been developed to guarantee that the closed-loop system enjoys the delay-dependent asymptotic stability with a prescribed $\gamma-$level $\mathcal{L}_2-$gain.

2. For the class of nominally-linear continuous-time systems with time-varying delays, new robust delay-dependent stability and stabilization methods have been established, for systems with unknown time-varying perturbations satisfying Lipschitz conditions. Again theorems for feedback and dynamic output controller have been established for such systems. In both schemes, the closed-loop feedback system has been shown to enjoy the delay-dependent asymptotic stability with a prescribed $\gamma-$level $\mathcal{L}_2-$gain. The feedback gains have been determined by convex optimization over LMIs.

3. New robust delay-dependent stability and stabilization methods for systems with interval delay type have been established. The theorem has been developed using no more than the required number of free-weighting matrices to build a fast and efficient method. The method has been found to be less conservative than the recent results in this type, while using fewer variables. Again the theorems have been formulated as LMIs.

All the developed methods and theorems have been verified and tested on representative examples through simulation. The simulation outcomes have enhanced the theoretically obtained results.

As a continuation of this work, the obtained results can be extended in different directions; each direction having its own application. The most important directions that can be followed are:

- **More reasonable conditions**: In this thesis, and in all the developed theorems in this field as well, there has been an assumption that ρ and μ can have their maximum values at the same instant. There is a contradiction in this assumption, because as the delay approaches its upper limit the rate of change should become less positive. In other words, as the delay increases, the rate of change should decrease, and when the delay reaches its maximum value the rate of change should have non-positive value as the delay cannot exceed the maximum limit. Thus, there is a need to consider the relation between ρ and μ at the same time. One solution is to consider the delay rate of change as a function of the delay value, so that better results may be obtained.

- **Probabilistic nature of the delay**: The second point is to consider the overall stability of the system with the time. As the delay varies, the system can be stable for some values of the delay and unstable for others. The system can be stable in a broad sense, depending on how much of the time the system is suffering destabilizing delay and how much it is not. Hence, good results may be obtained by using the probabilistic behavior of the delay in conjunction with the proposed methods.

- **Time delay in discrete time systems**: Finally, as most of the current systems are based on digital controllers, and as the direction in the industry is more toward networked control systems, it is more appropriate to look at the problem in the discrete time. The concepts behind the methods developed in this thesis can be used to find corresponding methods to deal with discrete time delay systems.

Nomenclature

$x(t)$: the state vector ($\in \Re^n$)

$y(t)$: the measured output vector($\in \Re^p$)

$u(t)$: the control input vector ($\in \Re^m$)

$w(t)$: the disturbance input vector ($\in \Re^q$)

$z(t)$: the controlled output vector ($\in \Re^q$)

t : time

τ : the time-delay factor

ϕ : a differentiable vector-valued function on $[-\tau, 0]$

A_o : the System matrix ($\in \Re^{nn}$)

A_{do} : tye delayed states matrix ($\in \Re^{nn}$)

B_o : the inputs matrix ($\in \Re^{nm}$)

G_o : the disturbance matrix ($\in \Re^{qn}$)

D_o : the matrix relates the inputs to tyhe controlled outputs ($\in \Re^{qm}$)

Φ_o : the matrix that relates the disturbances to the controlled outputs ($\in \Re^{qq}$)

Γ_o : the matrix that realtes the states to the controlled outputs ($\in \Re^{nq}$)

C_o : the matrix that relates the states to the outputs ($\in \Re^{pn}$)

C_{do} : the matrix that relates the delayed states to the outputs ($\in \Re^{pn}$)

F_o : the matrix that relates the inputs to the outputs ($\in \Re^{pm}$)

Ψ_o : the matrix that relates the disturbances to the outputs ($\in \Re^{pq}$)

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